2014 Senior External Examination

Mathematics B
Paper One — Question book

Monday 3 November 2014

Time allowed

• Perusal time: **10 minutes**
• Working time: **3 hours**

Examination materials provided

• Paper One — Question book
• Paper One — Resource book
• Paper One — Response book

Equipment allowed

• QCAA-approved equipment
• ruler graduated in millimetres
• protractor
• graphics calculator
• additional calculator

Equipment not allowed

• Calculators with computer algebra system (CAS) functionality

Directions

You may write in this book during perusal time.
Paper One has **six** questions. Attempt **all** questions.

Assessment

Paper One assesses the following assessment criteria:

• Knowledge and procedures (KP)
• Modelling and problem solving (MP)
• Communication and justification (CJ)

Assessment standards are at the end of this book.

After the examination session

Take this book when you leave.
Planning space
Question 1
Twenty-one parcels are weighed and their masses, in kilograms, are recorded in the following table.

<table>
<thead>
<tr>
<th>Mass in kg</th>
<th>Number of parcels</th>
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<tbody>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
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<tr>
<td>14</td>
<td>5</td>
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<td>15</td>
<td>6</td>
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</table>

a. Determine the five-number summary for this data distribution and present the data as a box-and-whisker plot (boxplot).

b. For the data distribution,
   i. Determine the mode.
   ii. Use a calculator to estimate the mean and standard deviation.
   iii. Construct the frequency histogram.

c. Data that is normally distributed has an area under the bell curve that is symmetrical about the mean; otherwise, the data is called asymmetrical. The measure of the asymmetry of a data distribution is called its skewness and this indicates the relationship between the measures of central tendency (mean, median and mode) and the shape of the distribution.

   In general:
   • in right-skewed data, the mode < median < mean
   • in left-skewed data, the mean < median < mode.

   The skewness, $S$, of a data distribution can be calculated by the formula

   \[ S = \frac{3 \text{(mean} - \text{median)}}{\text{standard deviation}} \]

   In a bi-modal distribution, this method of calculating skewness has a limitation.

   An additional parcel of mass 13 kg is added, making a total of 22 parcels. Examine the effects of this additional mass in calculating the skewness, $S$. 

(MP)
Question 2

a. For each of the following relations, state whether it is a function or not.

i. \{ (1, -4), (0, -3), (2, 5), (-3, 0), (-2, -4) \}

ii. 

\[ \begin{array}{c|c}
5 & -2 \\
3 & 4 \\
1 & 6 \\
0 & 7 \\
-1 & -9 \\
\end{array} \]

iii. 

\[ y \]
\[ \begin{array}{c|c}
5 & \end{array} \]
\[ \begin{array}{c|c}
0 & 5 \\
-5 & \end{array} \]

b. Determine the domain and range of the following function. 

\[ f(x) = \begin{cases} 
-\frac{8}{x+4}, & -4 < x \leq 0 \\
2x - 2, & 0 < x \leq 3 \\
4, & x > 3
\end{cases} \]

(KP)

c. Describe the set of transformations required to convert \( y = x^2 \) to \( y = 2x^2 - 4x + 5 \). 

(KP)

d. Tree seedlings in a nursery are grown until they reach 20 cm in height (a suitable height for sale). Two seedlings have been selected. The first seedling has been growing for three weeks and measures 9.3 cm in height. The second seedling has been growing for six weeks and measures 13.8 cm in height.

If plant growth is modelled by a linear equation, what height will the seedlings be when they are 12 weeks old?

Comment on any assumptions made and the effect these assumptions make to your solution. 

(MP)
Question 3

a. Using the Pythagorean identity, show that \(3 \cos \theta - 2 \sin^2 \theta + 1 = 2 \cos^2 \theta + 3 \cos \theta - 1\).  

(KP)

b. The water height in a creek fluctuates according to the tide. At time \(t\) hours after midnight, the height of the tide, \(h\) metres, is given by

\[
h = 1 + 0.6 \sin \frac{2\pi}{13} (t + 0.25)
\]

i. State the amplitude, period, horizontal and vertical shifts.

ii. Find the height of the tide at 6 am.

iii. Using an algebraic method, find when the tide is first at a height of 1 metre.

iv. Use a graphics calculator to graph \(h\) against \(t\). Determine the first time interval when the height of the water exceeds 1.5 metres.

(KP)

c. The angle of elevation from a point M on the ground to the top of a flagpole mounted on a building is 67° and to the top of the building is 65°. The building is 80 metres tall.

i. Calculate the distance from the point M to the base of the flagpole at B.

ii. Find the height of the flagpole correct to the nearest metre.

(KP)

d. Town X is N 15 °E of town Y at a distance of 1200 km. To avoid a thunderstorm, a pilot leaves town X and flies S 32 °W for 700 km and then directly on to town Y. How much further did the plane fly by not flying directly from town X to town Y?

(MP)
Question 4

a. Given that \( f(x) = 2x - 1 \) and \( g(x) = 3x^2 + x \), find:
   
i. \( g(-1) \)
   
ii. the inverse function, \( f^{-1}(x) \)
   
iii. the composite function, \( g(f(x)) \).  

b. Using algebraic methods, find the points of intersection of \( y = 2x^2 - 3x - 1 \) and \( y = x + 5 \). 

(c. A land developer subdivides a block of land into a number of smaller blocks. All blocks are sold for the same price, giving a total return of \( $1800000 \). Later it was calculated that, if the blocks had been made slightly larger, thereby reducing the total number of blocks for sale by one, and if the price had been increased by \( $50000 \) per block, then a total of \( $2000000 \) would have been returned from the sale.

Determine the number of blocks of land in the original subdivision.

Question 5

a. Show from first principles that the derivative of \( f(x) = 3x^2 - 1 \) is \( 6x \).

b. Determine \( \frac{dy}{dx} \) for each of the following:
   
i. \( y = x^3 + \frac{1}{x} - 5\log_e x + 4 \)
   
ii. \( y = \sqrt{2x + 1} \)
   
iii. \( y = \frac{\sin 4x}{x + 6} \)

(c. At time \( t \) seconds, the displacement of a particle is given as \( x \) metres where \( x = e^{t\log_e t} \).
   
i. Determine the velocity of the particle when \( t = 2 \) seconds.
   
ii. Calculate the average speed of the particle between \( t = 1 \) second and \( t = 3 \) seconds.
d. A horizontal gutter is to be made from a long sheet of iron 20 cm wide by turning one edge up vertically and the other edge in a semi-circle. The cross-section of the gutter is shown below.

Use calculus methods to show that, in order to maximise the capacity of the gutter, the radius of the semi-circle is given by

\[ r = \frac{40}{8 + 3\pi} \text{ cm} \]  

(MP)

Question 6

a. A box contains 12 blue marbles and 8 red marbles. A marble is randomly drawn from the box, its colour noted and then the marble put back in the box before another marble is drawn.

i. If two marbles are drawn from the box as described, what is the probability of obtaining a red marble and a blue marble in any order?

ii. If \( X \) is the random variable that represents the number of blue marbles drawn from the box in 14 trials, calculate \( P(X \geq 4) \).

(KP)

b. The blood pressure measurements of a group of women are normally distributed. The mean blood pressure is 129 and the standard deviation is 7.85. Within what blood pressure limits will the middle 80% of women fall?

(KP)

c. Three thousand electrical street lights are installed at the same time. The life span of these street lights is normally distributed with a mean of 1500 hours and a standard deviation of 100 hours. The manufacturer claims that fewer than 50 lights are expected to fail in the first 1300 hours. Examine the validity of this claim.

(MP)

End of Paper One
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<tr>
<td>Knowledge and procedures</td>
<td>The overall quality of a candidate’s achievement across the full range within the contexts of Application, Technology and Complexity, and across topics, <strong>consistently demonstrates:</strong> • accurate recall, selection and use of definitions and rules • accurate use of technology • recall and selection of procedures and their accurate and proficient use • effective transfer and application of mathematical procedures.</td>
<td>The overall quality of a candidate’s achievement across a range within the contexts of Application, Technology and Complexity, and across topics, <strong>generally demonstrates:</strong> • accurate recall, selection and use of definitions and rules • accurate use of technology • recall and selection of procedures and their accurate use.</td>
<td>The overall quality of a candidate’s achievement in the contexts of Application, Technology and Complexity <strong>generally demonstrates:</strong> • accurate recall and use of basic definitions and rules • use of technology • accurate recall, selection and use of basic procedures.</td>
<td>The overall quality of a candidate’s achievement in the contexts of Application, Technology and Complexity <strong>sometimes demonstrates:</strong> • accurate recall and use of some definitions and rules • use of technology • use of basic procedures.</td>
<td>The overall quality of a candidate’s achievement <strong>rarely demonstrates</strong> knowledge and use of procedures.</td>
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| **Modelling and problem solving** | The overall quality of a candidate’s achievement across the full range within each context, and across topics, generally demonstrates mathematical thinking which includes:  
• interpreting, clarifying and analysing a range of situations identifying assumptions and variables  
• selecting and using effective strategies  
• selecting suitable procedures required to solve a range of problems  
... and sometimes demonstrates mathematical thinking which includes:  
• suitable synthesis of procedures and strategies to solve problems  
• initiative and insight in exploring the problem  
• identifying strengths and limitations of models. | The overall quality of a candidate’s achievement across a range within each context, and across topics, generally demonstrates mathematical thinking which includes:  
• interpreting, clarifying and analysing a range of situations and identifying assumptions and variables  
• selecting and using effective strategies  
• selecting suitable procedures required to solve a range of problems  
... and sometimes demonstrates mathematical thinking which includes:  
• suitable synthesis of procedures and strategies to solve problems  
• initiative and insight in exploring the problem  
• identifying strengths and limitations of models. | The overall quality of a candidate’s achievement demonstrates mathematical thinking which includes:  
• interpreting and clarifying a range of situations  
• selecting strategies and/or procedures required to solve problems. | The overall quality of a candidate’s achievement sometimes demonstrates mathematical thinking which includes:  
• following basic procedures and/or using strategies. | The overall quality of a candidate’s achievement rarely demonstrates mathematical thinking which includes following basic procedures and/or using strategies. |
|---------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Communication and justification | The overall quality of a candidate’s achievement across the full range within each context consistently demonstrates:  
- accurate use of mathematical terms and symbols  
- accurate use of language  
- organisation of information into various forms suitable for a given use  
- use of mathematical reasoning to develop logical arguments in support of conclusions, results and/or propositions  
- justification of procedures  
- recognition of the effects of assumptions  
- evaluation of the validity of arguments. | The overall quality of a candidate’s achievement across a range within each context generally demonstrates:  
- accurate use of mathematical terms and symbols  
- accurate use of language  
- organisation of information into various forms suitable for a given use  
- use of mathematical reasoning to develop logical arguments in support of conclusions, results and/or propositions  
- justification of procedures. | The overall quality of a candidate’s achievement in all contexts generally demonstrates:  
- accurate use of basic mathematical terms and symbols  
- accurate use of language  
- organisation of information into various forms  
- use of some mathematical reasoning to develop simple logical arguments. | The overall quality of a candidate’s achievement sometimes demonstrates evidence of the basic conventions of language and mathematics and occasional use of mathematical reasoning. | The overall quality of a candidate’s achievement rarely demonstrates use of the basic conventions of language and mathematics. |