Specialist Mathematics marking guide

External assessment 2021

Paper 1: Technology-free (65 marks)

Paper 2: Technology-active (65 marks)

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from Units 3 and 4
2. comprehend mathematical concepts and techniques drawn from Units 3 and 4
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from Units 3 and 4.

Queensland Government | Queensland Curriculum & Assessment Authority

Purpose

The marking guide:

- provides a tool for calibrating external assessment markers to ensure reliability of results
- indicates the correlation, for each question, between mark allocation and qualities at each level of the mark range
- informs schools and students about how marks are matched to qualities in student responses.

Mark allocation

Where a response does not meet any of the descriptors for a question or a criterion, a mark of ‘0’ will be recorded.

Allowing for FT error — refers to ‘follow through’, where an error in the prior section of working is used later in the response, a mark (or marks) for the rest of the response can be awarded so long as it still demonstrates the correct conceptual understanding or skill in the rest of the response.

Allow FT mark/s — refers to ‘follow through’, where an error in the prior section of working is used later in the response, a mark (or marks) for the rest of the response can still be awarded so long as it still demonstrates the correct conceptual understanding or skill in the rest of the response.

This mark may be implied by subsequent working — the full mathematical reasoning and/or working, as outlined in the sample response and associated mark, is not explicitly stated in the student response, but by virtue of subsequent working there is sufficient evidence to award the mark/s.

Marking guide

Multiple choice
Paper 1

Question	Response
1	B
2	A
3	C
4	B
5	D
6	C
7	B
8	C
9	A
10	D

Short response

Q	Sample response	The response:
11a)	$f(2)=\tan^{-1}(1)$ $=\frac{\pi}{4}$ as $f(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	• correctly determines the required value **[1 mark]**
11b)	$f'(x)=\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right)$ $=\frac{2}{4+x^2}$ $f'(2)=\frac{1}{4}$	• correctly determines the gradient function **[1 mark]** • determines gradient of the tangent **[1 mark]**
11c)	Equation of the tangent has the form $y=mx+c$ From 11a) $x=2, y=\frac{\pi}{4}$ From 11b) $m=\frac{1}{4}$ $\frac{\pi}{4}=\frac{1}{4}(2)+c\Rightarrow c=\frac{\pi}{4}-\frac{1}{2}$ Equation of the tangent is $y=\frac{1}{4}x+\frac{\pi}{4}-\frac{1}{2}$	• determines y-intercept of the tangent **[1 mark]** • determines equation of the tangent **[1 mark]**

Q	Sample response	The response:
12a)	A vector perpendicular to $x - y - 2z = 15$ is $\boldsymbol{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$	• correctly determines a suitable vector $\boldsymbol{n}$ **[1 mark]**
12b)	Vector equation of line l is $\boldsymbol{r} = \boldsymbol{a} + k\boldsymbol{d}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $k \in R$	• determines vector equation of the line **[1 mark]**
12c)	Equation of line l in parametric form $x = -2 + k$ $y = 1 - k$ $z = 3 - 2k \quad k \in R$	• expresses equation of the line in parametric form **[1 mark]**
12d)	**Method 1** Given S lies on the plane $x - y - 2z = 15$ $(-2 + k) - (1 - k) - 2(3 - 2k) = 15$ $6k - 9 = 15 \Rightarrow 6k = 24$ $k = 4$ The coordinates of S are $(-2 + 4, 1 - 4, 3 - 8) = (2, -3, -5)$	• substitutes result from 12c) into the equation of the plane **[1 mark]** • determines value of the parameter **[1 mark]** • determines coordinates of S **[1 mark]**

Q	Sample response	The response:
	Method 2 Substituting $(2, -3, -5)$ into $x - y - 2z = 15$ $\text{LHS} = 2 - (-3) - 2(-5) = 15 = \text{RHS}$, so S lies on the plane.	• verifies that S lies on the plane **[1 mark]**
	Substituting $(2, -3, -5)$ into line l result from 12c) $2 = -2 + k \Rightarrow k = 4$	• uses x-coordinate of S to determine value of the parameter **[1 mark]**
	$-3 = 1 - k \Rightarrow k = 4$ $-5 = 3 - 2k \Rightarrow k = 4$ $\therefore$ S lies on line l	• shows that parameter value is consistent across the set of parametric equations by considering the remaining two coordinates of S **[1 mark]**
	Method 3 Substituting $(2, -3, -5)$ into $x - y - 2z = 15$ $\text{LHS} = 2 - (-3) - 2(-5) = 15 = \text{RHS}$, so S lies on the plane.	• verifies that S lies on the plane **[1 mark]**
	Substituting $(2, -3, -5)$ into line l result from 12b) $\begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $2 = -2 + k \Rightarrow k = 4$	• uses x-coordinate of S to determine the value of the parameter **[1 mark]**
	$-3 = 1 - k \Rightarrow k = 4$ $-5 = 3 - 2k \Rightarrow k = 4$ $\therefore$ S lies on line l	• shows that parameter value is consistent across the set of parametric equations by considering the remaining two coordinates of S **[1 mark]**

Q	Sample response	The response:
12e)	$\overrightarrow{AS} = s - a = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix}$	• determines $\overrightarrow{AS}$ **[1 mark]**
12f)	$\begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ ⇒ result is verified	• shows that $\overrightarrow{AS}$ is a scalar multiple of $\boldsymbol{n}$ **[1 mark]**
13	RTP $\lvert z-w\rvert^2 = \lvert z\rvert^2 + \lvert w\rvert^2 - 2Re(z\,\overline{w})$ $\text{LHS} = \lvert z-w\rvert^2$ $= \lvert(a+bi)-(c+di)\rvert^2$ $= \lvert(a-c)+(b-d)i\rvert^2$ $= (a-c)^2+(b-d)^2$ $= a^2-2ac+c^2+b^2-2bd+d^2$ $\text{RHS} = \lvert z\rvert^2 + \lvert w\rvert^2 - 2Re(z\,\overline{w})$ $= \lvert a+bi\rvert^2 + \lvert c+di\rvert^2 \ldots$ $\ldots -2Re\big((a+bi)(c-di)\big)$ $= a^2+b^2+c^2+d^2\ldots$ $\ldots -2Re\big((ac+bd)+(bc-ad)i\big)$	• correctly expresses $z-w$ in terms of a, b, c and d in Cartesian form **[1 mark]** • expresses $\lvert z-w\rvert^2$ in terms of a, b, c and d in expanded form **[1 mark]** • correctly expresses $\lvert z\rvert^2 + \lvert w\rvert^2$ in terms of a, b, c and d in expanded form **[1 mark]** • correctly expresses $(z\,\overline{w})$ in terms of a, b, c and d **[1 mark]**

Q	Sample response	The response:
	$= a^2 + b^2 + c^2 + d^2 - 2(ac + bd)$ $= a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$ $= a^2 - 2ac + c^2 + b^2 - 2bd + d^2$ $= \text{LHS}$	• completes proof **[1 mark]** • shows logical organisation, communicating key steps **[1 mark]**
14a)	**Method 1** $\boldsymbol{r}(t) = 5t(8 - t)\hat{\boldsymbol{j}}$ $= (40t - 5t^2)\,\hat{j}$ $\boldsymbol{v}(t) = (40 - 10t)\,\hat{\boldsymbol{j}}$	• correctly expands the given expression **[1 mark]** • determines $\boldsymbol{v}(t)$ **[1 mark]**
	Method 2 $\boldsymbol{r}(t) = 5t(8 - t)\hat{\boldsymbol{j}}$ Using the product rule $\boldsymbol{v}(t) = \big(5t(-1) + 5(8 - t)\big)\hat{j}$ $= (-5t + 40 - 5\text{t})\hat{\boldsymbol{j}}$ $= (40 - 10t)\hat{\boldsymbol{j}}$	• correctly uses the product rule to determine $\boldsymbol{v}(t)$ **[1 mark]** • determines $\boldsymbol{v}(t)$ **[1 mark]**
14b)	Maximum height occurs when $\boldsymbol{v}(t) = 0\hat{\boldsymbol{j}}$ $40 - 10t = 0$ $10t = 40$ $t = 4\text{ s}$	• establishes equation in terms of t **[1 mark]** • determines t **[1 mark]**

Q	Sample response	The response:
14c)	Maximum height $= 5 \times 4(8-4)$ $= 20 \times 4$ $= 80$ m	• substitutes result from 14b) into expression for position **[1 mark]** • determines maximum height **[1 mark]**
15	$\dfrac{4x-17}{x^2-x-6} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-3)}$ $= \dfrac{A(x-3)+B(x+2)}{(x+2)(x-3)}$ $x = 3\colon\ -5 = 5B \Rightarrow B = -1$ $x = -2\colon\ -25 = -5A \Rightarrow A = 5$ $\displaystyle\int \frac{4x-17}{x^2-x-6}\,dx = \int \frac{5}{(x+2)} + \frac{-1}{(x-3)}\ dx$ $= 5\ln\lvert x+2\rvert - \ln\lvert x-3\rvert$ $= \ln\lvert x+2\rvert^5 - \ln\lvert x-3\rvert$ $= \ln\left\lvert\dfrac{(x+2)^5}{x-3}\right\rvert + c$	• correctly factorises the denominator to establish the form of the partial fraction decomposition **[1 mark]** • determines values of A and B **[1 mark]** • determines indefinite integral of the fraction **[1 mark]** • determines expression in the form $\ln\lvert f(x)\rvert$ **[1 mark]**

Q	Sample response	The response:
16	RTP $2^{2n}+3n-1$ is always divisible by 3 for $n \in Z^+$, i.e. $2^{2n}+3n-1=3m, \text{for some } m \in Z^+$ Initial statement: Let $n=1$: $2^2+3-1=6$ $=3\times 2$ Proposition is true for $n=1$	• correctly proves the initial statement **[1 mark]**
	Assume proposition is true for $n=k \ \forall \ k \in Z^+$ $2^{2k}+3k-1=3m$ for some $m \in Z^+$	• correctly formulates an assumption for $n=k$ **[1 mark]**
	Inductive step: Let $n=k+1$ $2^{2(k+1)}+3(k+1)-1=2^{2k+2}+3k+3-1$ $=2^2\times 2^{2k}+3k-1+3$	• correctly establishes an expression representing the LHS of the inductive step proof and uses an index law to establish a term with a factor of 2^{2k} **[1 mark]**
	$=2^{2k}+3k-1+3\times 2^{2k}+3$ $=3m+3\times 2^{2k}+3$	• uses previous assumption in the inductive step **[1 mark]**
	$=3(m+2^{2k}+1)$ $=3p$ for some $p \in Z^+$	• completes proof by determining an expression with a factor of 3 representing the RHS of the inductive step proof **[1 mark]**
	The proposition is true for $n=k+1$ By mathematical induction, the proposition is true for $n=1,2,...$	• shows logical organisation, having attempted all steps of the proof, including the use of a suitable conclusion **[1 mark]**

Q	Sample response	The response:
17	Finding the points of intersection of the two functions $y = 4x$ and $y = 2x^2$ $4x = 2x^2$ $2x^2 - 4x = 0 \Rightarrow 2x(x-2) = 0 \Rightarrow x = 0$ and $x = 2$ When $x = 0, y = 0$ When $x = 2, y = 8$	• correctly uses simultaneous equations to establish an equation in one unknown **[1 mark]** • correctly determines y-coordinates of the points of intersection **[1 mark]**
	Rearranging the two functions in the form $x = f(y)$ and $x = g(y)$ $x = \frac{y}{4}$ and $x = \pm\sqrt{\frac{y}{2}}$	• correctly determines functions in the form $x = f(y)$ **[1 mark]**
	Finding volume of revolution between curves $V = \left\lvert \pi \int_a^b [f(y)]^2 - [g(y)]^2 \, dy \right\rvert$ $= \left\lvert \pi \int_0^8 \frac{y}{2} - \frac{y^2}{16} \, dy \right\rvert$	• determines expression to represent the volume between the two curves **[1 mark]**
	$= \pi \left\lvert \frac{y^2}{4} - \frac{y^3}{48} \Big\rvert_0^8 \right\rvert$	• integrates expression **[1 mark]**
	$= \pi \left\lvert \left(16 - \frac{32}{3}\right) - (0) \right\rvert$ $= \frac{16\pi}{3}$	• determines (positive) value of V in terms of π **[1 mark]**
		• shows logical organisation, communicating key steps to at least the start of finding the volume of revolution **[1 mark]**

Q	Sample response	The response:
18	$k\frac{dI}{dt} + RI = V \Rightarrow k\frac{dI}{dt} = V - RI$ $\int \frac{k}{V-RI}\,dI = \int 1\,dt$	• correctly uses the separation of variables method to set up indefinite integrals **[1 mark]**
	$-\frac{k}{R}\ln\lvert V-RI\rvert = t + c$ Given $I = 0$ when $t = 0$	• develops a general solution of the differential equation **[1 mark]**
	$c = -\frac{k}{R}\ln(V) \qquad (\text{as } V > 0)$	• uses the given condition to determine expression for the constant of integration **[1 mark]**
	$-\frac{k}{R}\ln\lvert V-RI\rvert = t - \frac{k}{R}\ln(V)$ $\ln\lvert V-RI\rvert = -\frac{R}{k}t + \ln(V)$	• rearranges relationship to express $\ln\lvert V - RI\rvert$ as the subject of the equation **[1 mark]**
	$V - RI = e^{-\frac{R}{k}t + \ln(V)}$ $V - RI = V\,e^{-\frac{R}{k}t}$	• expresses relationship as an exponential function **[1 mark]**
	For all t, $e^{-\frac{R}{k}t} > 0 \Rightarrow V - RI > 0 \Rightarrow V > RI$ $\Rightarrow I < \frac{V}{R}$ So, the size of the current can never be greater than $\frac{V}{R}$.	• considers value of I over time to determine the required limit **[1 mark]**

Q	Sample response	The response:
19	$\boldsymbol{v}_A = 6\sin(3t)\hat{\boldsymbol{\imath}} + 6\cos(3t)\hat{\boldsymbol{\jmath}}$ $\boldsymbol{v}_B = \cos(t)\hat{\boldsymbol{\imath}} - \sin(t)\hat{\boldsymbol{\jmath}}$ $\boldsymbol{r}_A = \int \boldsymbol{v}_A\,dt = -2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + c_A$ When $t = 0$ $-2\hat{\imath} + 2\hat{\boldsymbol{k}} = -2\cos(0)\hat{\boldsymbol{\imath}} + 2\sin(0)\hat{\boldsymbol{\jmath}} + \boldsymbol{c}_A \Rightarrow \boldsymbol{c}_A = 2\hat{\boldsymbol{k}}$ $\therefore \boldsymbol{r}_A = -2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$	• correctly determines the expression for the position of Object A **[1 mark]**
	$\boldsymbol{r}_B = \int \boldsymbol{v}_B\,dt = \sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} + \boldsymbol{c}_B$ When $t = 0$ $\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}} = \sin(0)\hat{\boldsymbol{\imath}} + \cos(0)\hat{\boldsymbol{\jmath}} + \boldsymbol{c}_B \Rightarrow \boldsymbol{c}_B = -\hat{\boldsymbol{k}}$ $\therefore \boldsymbol{r}_B = \sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$	• correctly determines the expression for the position of Object B **[1 mark]**
	$\boldsymbol{r}_B - \boldsymbol{r}_A$ $= \left(\sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}\right)\ldots$ $\ldots -\left(-2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}\right)$ $= (\sin(t) + 2\cos(3t))\hat{\boldsymbol{\imath}} + (\cos(t) - 2\sin(3t))\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$	• determines an expression to represent the relative position of Objects A and B **[1 mark]**
	$\lvert \boldsymbol{r}_B - \boldsymbol{r}_A \rvert$ $= \sqrt{\sin^2(t) + 4\sin(t)\cos(3t) + 4\cos^2(3t) + \cdots}$ $\overline{\ldots\cos^2(t) - 4\cos(t)\sin(3t) + 4\sin^2(3t) + 9}$ $= \sqrt{14 - 4(\sin(3t)\cos(t) - \cos(3t)\sin(t))}$ $= \sqrt{14 - 4(\sin(3t - t))}$	• determines an expression to represent the distance (or square of the distance) between the objects **[1 mark]**

Q	Sample response	The response:
	$= \sqrt{14 - 4\sin(2t)}$ Given $\lvert \boldsymbol{r}_B - \boldsymbol{r}_A \rvert = 4$ $\sqrt{14 - 4\sin(2t)} = 4$ $\sin(2t) = -\frac{1}{2}$ $2t = \frac{7\pi}{6}$ $t = \frac{7\pi}{12}$ s (first positive solution) Position of A $\boldsymbol{r}_A = -2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ $= -2\cos\left(\frac{7\pi}{4}\right)\hat{\boldsymbol{\imath}} + 2\sin\left(\frac{7\pi}{4}\right)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ $= -\sqrt{2}\hat{\boldsymbol{\imath}} - \sqrt{2}\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ (m)	• uses a trigonometric identity to determine an expression in terms of a single trigonometric function that represents the distance (or square of the distance) between the objects **[1 mark]** • determines the first time that Object A is 4 metres away from Object B **[1 mark]** • determines position of Object A **[1 mark]**

Marking guide

Multiple choice
Paper 2

Question	Response
1	C
2	D
3	A
4	C
5	D
6	A
7	B
8	B
9	C
10	B

Short response

Q	Sample response	The response:
11	$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix}$ $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$	· correctly determines the two vectors representing any two sides of the triangle ABC **[1 mark]**
	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 21 \\ 1 \\ -6 \end{pmatrix}$	· determines cross product of two vectors representing any two sides of the triangle ABC **[1 mark]**
	Area $= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix} 21 \\ 1 \\ -6 \end{pmatrix}\right\|$ $= \frac{1}{2}\sqrt{21^2 + 1^2 + (-6)^2}$	· uses vector expression to represent the area of a triangle **[1 mark]**
	$= 10.93 \text{ units}^2$	· determines area **[1 mark]**

Q	Sample response	The response:
12a)	$\overline{x} = \frac{2321.4+2423.6}{2} = 2372.5$ hours	· correctly determines the mean lifetime of the sample **[1 mark]**
12b)	Approximate confidence interval for μ is $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$ Using the upper value of the 95% confidence interval $2372.5 + 1.96\frac{125}{\sqrt{n}} = 2423.6$ Solving equation: $n \approx 22.98$ Sample size used was 23.	· determines an equation in terms of n **[1 mark]** · solves equation for n **[1 mark]** · expresses sample size rounded to a whole number **[1 mark]**
12c)	Yes, the assumption is required. Because the sample size is less than 30, the distribution of the sample mean cannot be assumed to be normal unless the population is normally distributed.	· makes a suitable statement **[1 mark]** · discusses the sample size and its relation to a normal distribution **[1 mark]**

<table>
<tr><th>Q</th><th>Sample response</th><th>The response:</th></tr>
<tr><td>13a)</td><td>Given $a = 1,\ b = 9,\ n = 4$
$w = \dfrac{b-a}{n} = 2$

Table of values
<table><tr><td>x</td><td>$f(x) = 0.2e^{-0.2x}$</td></tr><tr><td>1</td><td>$f(1) = 0.2e^{-0.2} = 0.164$</td></tr><tr><td>3</td><td>$f(3) = 0.2e^{-0.6} = 0.110$</td></tr><tr><td>5</td><td>$f(5) = 0.2e^{-1} = 0.074$</td></tr><tr><td>7</td><td>$f(7) = 0.2e^{-1.4} = 0.049$</td></tr><tr><td>9</td><td>$f(9) = 0.2e^{-1.8} = 0.033$</td></tr></table>
Using Simpson's rule:
Area » $\frac{2}{3}\Big(f(1) + 4\big(f(3) + f(7)\big) + 2\big(f(5)\big) + f(9)\Big)$

» $\dfrac{2}{3}(0.9803)$
$\approx 0.65\ \text{units}^2$</td><td>· correctly determines the interval width required for the calculations **[1 mark]**

· correctly determines all required values of $f(x)$ **[1 mark]**

· substitutes values into Simpson's rule **[1 mark]**

· approximates the area using Simpson's rule **[1 mark]**</td></tr>
<tr><td>13b)</td><td>Area $= \int_1^9 0.2e^{-0.2x}\ dx$

$\approx 0.65\ \text{units}^2$
The approximation is reasonable.</td><td>· correctly expresses the exact area as a definite integral **[1 mark]**

· calculates the value of the definite integral and provides a suitable comment on the reasonableness of the response **[1 mark]**</td></tr>
</table>

Q	Sample response	The response:
14a)	$x = 0.25$ $y = 0.3$	· correctly states the values of x and y **[1 mark]**
14b)	Number of females at the start of 2021 $\mathbf{N}_0 = \begin{bmatrix} 510 \\ 480 \\ 420 \end{bmatrix}$ Number of females at the start of 2025 $\mathbf{N}_4 = \mathbf{L}^4\,\mathbf{N}_0$ $= \begin{bmatrix} 2166 \\ 938.25 \\ 81.45 \end{bmatrix}$ Total female population $\approx 2166 + 938 + 81$ ≈ 3185 Total population $\approx \frac{3}{2} \times 3186 \approx 4778$	· correctly identifies the initial matrix **[1 mark]** · determines female population in their first, second and third years at the start of 2025 **[1 mark]** · determines total female population at the start of 2025 **[1 mark]** · determines total population at the start of 2025, rounded to a whole number **[1 mark]**

Q	Sample response	The response:
15a)	$V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $= \frac{\pi h^3}{12}$	· correctly determines a function for the volume in simplified form **[1 mark]**
15b)	$\frac{\boldsymbol{dV}}{\boldsymbol{dh}} = \frac{\boldsymbol{\pi h^2}}{\boldsymbol{4}}$ $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$ $2 = \frac{\pi h^2}{4}\frac{dh}{dt}$ $\frac{dh}{dt} = \frac{8}{\pi h^2}$	· determines an expression for $\frac{dV}{dh}$ **[1 mark]** · correctly states a related rates equation in terms of V, h and t **[1 mark]** · demonstrates suitable working leading to the required result **[1 mark]**
15c)	Half capacity $= \frac{1}{2}\left(\frac{1}{3}\boldsymbol{\pi r^2 h}\right) = \frac{1}{6}\boldsymbol{\pi(6)^2(12)}$ $\approx \mathbf{226.195\ cm^3}$ Solving $\frac{\boldsymbol{\pi h^3}}{\mathbf{12}} = \mathbf{226.195}$ $\therefore \boldsymbol{h} \approx \mathbf{9.52}$ cm $\frac{dh}{dt} \approx \frac{8}{\pi(9.52)^2} \approx 0.03$ Required rate is 0.03 cm s^{-1}	· correctly uses a suitable equation to determine h when the cup is half full **[1 mark]** · determines h when V reaches half capacity **[1 mark]** · determines required rate **[1 mark]** · correctly communicates relevant units **[1 mark]**

Q	Sample response	The response:
16	Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ $\mathbf{A}^4 = p\mathbf{A}^3 + q\mathbf{A}^2 + r\mathbf{A} + 3\mathbf{I}$ Using GDC and substituting into given equation $\begin{bmatrix} 177 & 176 & 144 \\ 144 & 145 & 120 \\ 176 & 176 & 145 \end{bmatrix} = p\begin{bmatrix} 37 & 38 & 32 \\ 32 & 31 & 25 \\ 38 & 38 & 31 \end{bmatrix} + q\begin{bmatrix} 9 & 8 & 6 \\ 6 & 7 & 6 \\ 8 & 8 & 7 \end{bmatrix}$ $+ r\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	· correctly substitutes results into given equation to form an equation based on 3 × 3 matrices **[1 mark]**
	Equating elements in Row 1: $177 = 37p + 9q + r + 3 \Rightarrow 37p + 9q + r = 174$ $176 = 38p + 8q + 2r + 0 \Rightarrow 38p + 8q + 2r = 176$ $144 = 32p + 6q + 2r + 0 \Rightarrow 32p + 6q + 2r = 144$	· uses equation based on 3 × 3 matrices to establish 3 equations in 3 unknowns **[1 mark]**
	Expressing the 3 equations in matrix form: $\begin{bmatrix} 37 & 9 & 1 \\ 38 & 8 & 2 \\ 32 & 6 & 2 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 174 \\ 176 \\ 144 \end{bmatrix}$	· expresses 3 equations in matrix form **[1 mark]**
	$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 37 & 9 & 1 \\ 38 & 8 & 2 \\ 32 & 6 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 174 \\ 176 \\ 144 \end{bmatrix}$	· demonstrates the use of matrix algebra in solving the matrix equation **[1 mark]**
	$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$	· determines values of p, q and r **[1 mark]**
		· shows logical organisation, communicating key steps to at least the start of expressing the 3 equations in matrix form **[1 mark]**

Q	Sample response	The response:
17	**Method 1** Resolving net forces along the plane $F_{\text{net}} = 2g\sin(30°) - \dfrac{4}{\sqrt{4-x^2}}$ $F_{\text{net}} = ma$ $g - \dfrac{4}{\sqrt{4-x^2}} = 2a$ $g - \dfrac{4}{\sqrt{4-x^2}} = 2v\,\dfrac{dv}{dx}$ $v\,\dfrac{dv}{dx} = 4.9 - \dfrac{2}{\sqrt{4-x^2}}$ $\int v\ dv = \int 4.9 - \dfrac{2}{\sqrt{4-x^2}}\ dx$ $\dfrac{v^2}{2} = 4.9x - 2\sin^{-1}\left(\dfrac{x}{2}\right) + c$ Given $v = 0$ when $x = 0$ $0 = 0 - 2\sin^{-1}(0) + c$ $c = 0$ $\therefore v^2 = 9.8x - 4\sin^{-1}\left(\dfrac{x}{2}\right)$ When $v = 2$ $4 = 9.8x - 4\sin^{-1}\left(\dfrac{x}{2}\right)$ Solving for x using GDC $x = 0.51$ m	· correctly determines the net forces along the plane **[1 mark]** · determines equation for acceleration along the plane **[1 mark]** · determines differential equation in terms of velocity and displacement **[1 mark]** · determines general solution to a differential equation **[1 mark]** · determines value of arbitrary constant **[1 mark]** · establishes equation to solve for x when $v = 2$ **[1 mark]** · determines x **[1 mark]**

Q	Sample response	The response:
17	**Method 2** Resolving net forces along the plane $F_{\text{net}} = 2g\sin(30°) - \frac{4}{\sqrt{4-x^2}}$ $F_{\text{net}} = ma$ $g - \frac{4}{\sqrt{4-x^2}} = 2a$	· correctly determines the net forces along the plane **[1 mark]** · determines equation for acceleration along the plane **[1 mark]**
	$g - \frac{4}{\sqrt{4-x^2}} = 2v\,\frac{dv}{dx}$ $v\,\frac{dv}{dx} = 4.9 - \frac{2}{\sqrt{4-x^2}}$ $\int v\ dv = \int 4.9 - \frac{2}{\sqrt{4-x^2}}\ dx$ $\frac{v^2}{2} = 4.9x + 2\cos^{-1}\left(\frac{x}{2}\right) + c$	· determines differential equation in terms of velocity and displacement **[1 mark]** · determines general solution to a differential equation **[1 mark]**
	Given $v = 0$ when $x = 0$ $0 = 0 + 2\cos^{-1}(0) + c$ $c = -\pi$ $\therefore v^2 = 9.8x + 4\cos^{-1}\left(\frac{x}{2}\right) - 2\pi$	· determines value of arbitrary constant **[1 mark]**
	When $v = 2$ $4 = 9.8x + 4\cos^{-1}\left(\frac{x}{2}\right) - 2\pi$	· establishes equation to solve for x when $v = 2$ **[1 mark]**
	Solving for x using GDC $x = 0.51$ m	· determines x **[1 mark]**

Q	Sample response	The response:
18	**Method 1** The roots of $z^5 = 1$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k \in Z$ Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the four roots of $z^4 + z^3 + z^2 + z + 1$ must be the four complex roots of the five 5th roots of unity, $z^5 = 1$. By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5 = 1$. One possible pair of roots is $\text{cis}\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(-\frac{2\pi}{5}\right)$ Determining a quadratic factor of $P(z)$ $\left(z - \text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$ $= z^2 - \left(\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)\right)z + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)$ $= z^2 - \left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)\right)z + \text{cis}(0)$ $= z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1$ $P(z) = (z-2)\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)$ $= z^3 - 2\left(\cos\left(\frac{2\pi}{5}\right) + 1\right)z^2 + \left(4\cos\left(\frac{2\pi}{5}\right) + 1\right)z - 2$	· correctly determines the roots of $z^5 = 1$ **[1 mark]** · correctly recognises one possible pair of roots **[1 mark]** · determines a quadratic factor of $P(z)$ in factorised form **[1 mark]** · expresses determined quadratic factor of $P(z)$ in expanded form **[1 mark]** · uses the factor of $z = 2$ to express $P(z)$ in factorised form **[1 mark]** determines $P(z)$ in expanded form **[1 mark]**

Q	Sample response	The response:
18	**Method 2** The roots of $z^5 = 1$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k \in Z$ Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the four roots of $z^4 + z^3 + z^2 + z + 1$ must be the four complex roots of the five 5th roots of unity, $z^5 = 1$. By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5 = 1$. One possible pair of roots is $\text{cis}\left(\frac{4\pi}{5}\right)$ and $\text{cis}\left(-\frac{4\pi}{5}\right)$. Determining a quadratic factor of $P(z)$ $\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$ $= z^2 - \left(\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)$ $= z^2 - \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}(0)$ $= z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1$ $P(z) = (z-2)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$ $= z^3 - 2\left(\cos\left(\frac{4\pi}{5}\right) + 1\right)z^2 + \left(4\cos\left(\frac{4\pi}{5}\right) + 1\right)z - 2$	· correctly determines the roots of $z^5 = 1$ **[1 mark]** · correctly determines one possible pair of roots **[1 mark]** · determines a quadratic factor of $P(z)$ in factorised form **[1 mark]** · expresses determined quadratic factor of $P(z)$ in expanded form **[1 mark]** · uses the factor of $z = 2$ to express $P(z)$ in factorised form **[1 mark]** · determines $P(z)$ in expanded form **[1 mark]**

Q	Sample response	The response:
18	**Method 3** Given $z = 2$ is a root of $P(z)$ $\therefore P(2) = 0 \Rightarrow (2)^3 + a(2)^2 + 2b + c = 0$ $8 + 4a + 2b + c = 0$ …(1)	· correctly determines equation using $z = 2$ **[1 mark]**
	The roots of $z^5 = 1$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k \in Z$ Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the four roots of $z^4 + z^3 + z^2 + z + 1$ must be the four complex roots of the five 5th roots of unity, $z^5 = 1$.	· correctly determines the roots of $z^5 = 1$ **[1 mark]**
	By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5 = 1$. One possible pair of roots is $\text{cis}\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(-\frac{2\pi}{5}\right)$	· determines one possible pair of roots **[1 mark]**
	$P(z) = z^3 + az^2 + bz + c$ $P\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) = 0$ $\therefore \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^3 + a\left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^2 + b\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) + c = 0$ $\text{cis}\left(\frac{6\pi}{5}\right) + a\,\text{cis}\left(\frac{4\pi}{5}\right) + b\,\text{cis}\left(\frac{2\pi}{5}\right) + c = 0$ …(2) $P\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right) = 0$ $\therefore \left(\text{cis}\left(-\frac{2\pi}{5}\right)\right)^3 + a\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right)^2 + \ldots$ $\ldots\, b\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right) + c = 0$ $\text{cis}\left(-\frac{6\pi}{5}\right) + a\,\text{cis}\left(-\frac{4\pi}{5}\right) + b\,\text{cis}\left(-\frac{2\pi}{5}\right) + c = 0$ …(3)	· determines two equations using two possible conjugate pairs **[1 mark]**
	Solving the three simultaneous equations using GDC $a \approx -2.62, b \approx 2.236, c = -2$	· solves the simultaneous equations to determine a, b and c **[1 mark]**
	$P(z) \approx z^3 - 2.62z^2 + 2.24z - 2$	· determines $P(z)$ in expanded form **[1 mark]**

Q	Sample response	The response:
19	Using the property of a PDF $\int_{-\infty}^{\infty} p(x)dx = 1$ Using $n = 5$ in the given PDF $\int_0^{\infty} \frac{k^5\, t^4}{4!}\, e^{-\frac{t}{3}}\, dt = 1$ Solving the equation: $k = \frac{1}{3}$ Mean of distribution for waiting time until 5th call, μ $E(X) = \int_{-\infty}^{\infty} x\, p(x)dx$ $\mu = \int_0^{\infty} t\, \frac{\left(\frac{1}{3}\right)^5 t^4}{4!}\, e^{-\frac{t}{3}}\, dt = \int_0^{\infty} \frac{\left(\frac{1}{3}\right)^5 t^5}{4!}\, e^{-\frac{t}{3}}\, dt$ = 15 minutes Variance of distribution for 5th call $Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx$ $= \int_0^{\infty} (t-15)^2 \frac{\left(\frac{1}{3}\right)^5 t^4}{4!} e^{-\frac{t}{3}} dt = 45 \text{ minutes}^2$ $\therefore \sigma = \sqrt{45}$ minutes Consider the distribution of the sample mean of the waiting time until the 5th phone call is received, $\overline{T}$. As the sample size is large, the distribution of $\overline{T}$ can be considered normal. $\mu_{\overline{T}} = 15 \text{ and } \sigma_{\overline{T}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{45}}{\sqrt{80}} = 0.75$ Using normal cdf on GDC: $P(\overline{T} > 16) \approx 0.09$	· correctly determines equation in terms of k **[1 mark]** · solves equation to determine k **[1 mark]** · determines population mean **[1 mark]** · determine population variance **[1 mark]** · justifies that the distribution of $\overline{T}$ can be considered normal **[1 mark]** · determines mean and standard deviation of the sample mean [**1 mark]** · determines required probability **[1 mark]**