Specialist Mathematics 2019 v1.2

IA3 sample marking scheme

August 2019

Examination (15%)

This sample has been compiled by the QCAA to model one possible approach to allocating marks in an examination. It matches the examination mark allocations as specified in the syllabus (~ 60% simple familiar, ~ 20% complex familiar and ~ 20% complex unfamiliar) and ensures that all assessment objectives are assessed.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 4 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 4 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions, and prove propositions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 4 topics.

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Instrument-specific marking guide (ISMG)

Criterion: Foundational knowledge and problem-solving

Assessment objectives

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 4 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 4 topics
3. communicate using mathematical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 4 topics.

The student work has the following characteristics:	Cut-off	Marks
• consistently correct selection, recall and use of facts, rules, definitions and procedures; authoritative and accurate command of mathematical concepts and techniques; astute evaluation of the reasonableness of solutions and use of mathematical reasoning to correctly justify procedures and decisions, and prove propositions; and fluent application of mathematical concepts and techniques to solve problems in a comprehensive range of simple familiar, complex familiar and complex unfamiliar situations.	> 93%	15
	> 87%	14
• correct selection, recall and use of facts, rules, definitions and procedures; comprehension and clear communication of mathematical concepts and techniques; considered evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions, and prove propositions; and proficient application of mathematical concepts and techniques to solve problems in simple familiar, complex familiar and complex unfamiliar situations.	> 80%	13
	> 73%	12
• thorough selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions, and prove propositions; and application of mathematical concepts and techniques to solve problems in simple familiar and complex familiar situations.	> 67%	11
	> 60%	10
• selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of some solutions using mathematical reasoning; and application of mathematical concepts and techniques to solve problems in simple familiar situations.	> 53%	9
	> 47%	8
• some selection, recall and use of facts, rules, definitions and procedures; basic comprehension and communication of mathematical concepts and techniques; inconsistent evaluation of the reasonableness of solutions using mathematical reasoning; and inconsistent application of mathematical concepts and techniques.	> 40%	7
	> 33%	6
• infrequent selection, recall and use of facts, rules, definitions and procedures; basic comprehension and communication of some mathematical concepts and techniques; some description of the reasonableness of solutions; and infrequent application of mathematical concepts and techniques.	> 27%	5
	> 20%	4
• isolated selection, recall and use of facts, rules, definitions and procedures; partial	> 13%	3

The student work has the following characteristics:	Cut-off	Marks
comprehension and communication of rudimentary mathematical concepts and techniques; superficial description of the reasonableness of solutions; and disjointed application of mathematical concepts and techniques.	> 7%	2
• isolated and inaccurate selection, recall and use of facts, rules, definitions and procedures; disjointed and unclear communication of mathematical concepts and techniques; and illogical description of the reasonableness of solutions.	> 0%	1
• does not satisfy any of the descriptors above.		0

Task

See the sample assessment instrument for IA3: Examination — short response (15%) available on the QCAA Portal.

Sample marking scheme

Criterion	Marks allocated	Result
Foundational knowledge and problem-solving Assessment objectives 1, 2, 3, 4, 5 and 6	15	
Total	**15**	

The annotations are written descriptions of the expected response for each question and are related to the assessment objectives.

Note: ✓ = $\frac{1}{2}$ mark

Paper 1 (technology-free)

1.

recall and use:

- appropriate setup of substitution ✓
- substitution into the integral ✓
- simplification of integrand ✓

select appropriate antiderivative rule ✓

recall process to express antiderivative in terms of x (for any constant value c) ✓

communicate by organising information using mathematical terminology, symbols and conventions ✓

Question 1 (3 marks) SF

Let $u = x^2 + 3$

$\frac{du}{dx} = 2x$ ✓

$\int 2xe^{x^2+3}\ dx = \int \frac{du}{dx} \cdot e^u\, dx$ ✓

$= \int e^u\, du$ ✓

$= e^u$ ✓ $+ c$

$= e^{x^2+3}$ ✓

✓

Question 2 (4 marks) SF

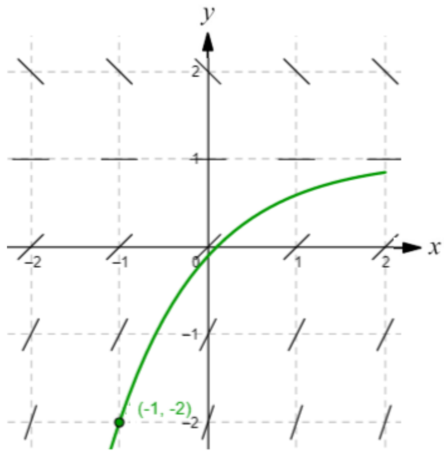
a. Given $\frac{dy}{dx} = 1 - y$

y	2	1	0	–1	–2
$\frac{dy}{dx}$	–1	0	1	2	3

2.

recall and use procedure to sketch the slope field by considering the gradient across the range of $-2 \leq y \leq 2$ (using table or otherwise) ✓✓✓

Sketch of slope field ✓✓✓✓✓✓



communicate by accurately presenting information in graphical form ✓✓✓

sketch an appropriate solution curve (allow x-intercept tolerance of ± 0.5) ✓✓

b. Solution curve from (–1, –2) (see green curve) ✓✓

3a.

comprehend the need to use integration by parts ✓

select and use procedure to integrate function ✓✓✓✓✓

Question 3 (3, 2 marks) SF

a. Using integration by parts to find $\int 4x \ln(x)\, dx$ ✓

Let $u = \ln(x)$ $\frac{dv}{dx} = 4x$

$\frac{du}{dx} = \frac{1}{x}$ ✓ $v = 2x^2$ ✓

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

$$\int \ln(x)\ 4x\ \ dx\ = 2x^2 \ln(x) - \int 2x^2 \tfrac{1}{x}\, dx \quad ✓✓$$

$$= 2x^2 \ln(x) - \int 2x\, dx$$

$$= \ 2x^2 \ln(x) - \ x^2 + c \quad ✓$$

3b.

recall procedure to:

- substitute limits of integration ✓✓
- simplify to required result ✓

communicate by organising information using mathematical terminology, symbols, conventions and everyday language ✓

b. Shaded area $= \int_1^e 4x \ln(x)\, dx$

$$= \ [2x^2 \ln(x) - \ x^2]_1^e \quad ✓$$

$$= \ (2e^2 \ln(e) - \ e^2) - (0 - 1) ✓$$

$$= \ (2e^2 - \ e^2 + 1)$$

$$= \ e^2 + 1 ✓$$

✓

Question 4 (5 marks) SF

4.

select and use:

- rule to find the volume of revolution of the given function about the x-axis ✓
- appropriate identity to find integral ✓✓

recall procedure to:

- find antiderivatives of each integrand function ✓✓

Volume of revolution $= \pi \int_a^b y^2 dx$

$$= \pi \int_0^{\frac{\pi}{4}} (\cos(x))^2\, dx \quad ✓$$

$$= \pi \int_0^{\frac{\pi}{4}} \cos^2(x)\, dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} (1 + \cos(2x))\, dx \quad ✓✓$$

$$= \frac{\pi}{2}\left[x + \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}}$$ ✓✓

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right]$$ ✓

$$= \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right]$$ ✓

$$= \frac{\pi(\pi+2)}{8}$$ ✓ (units^3) ✓✓

- substitute limits of integration ✓
- recall exact trigonometric values ✓
- simplify result ✓

communicate by organising information using mathematical terminology, symbols, conventions ✓✓

5.

select appropriate rule ✓

comprehend:

- to find derivative of V with respect to r ✓
- the definition of rate of change of r with respect to t ✓

recall procedure to:

- correctly substitute into the chain rule to find the derivative of V with respect to t ✓✓
- correctly find the rate of increase of the volume at $r = 5$ ✓

communicate by organising using mathematical terminology (suitable units), conventions and everyday language ✓✓

Question 5 (4 marks) SF

$V = \frac{4}{3}\pi r^3$ ✓ $(r \geq 4)$

$\therefore \frac{dV}{dr} = 4\pi r^2$ ✓

Given $\frac{dr}{dt} = 3$ ✓$(\mathrm{cm/s})$

Using chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot 3$$ ✓

$$= 12\pi r^2$$ ✓

When $r = 5$

$$\frac{dV}{dt} = 12\,\pi \cdot 5^2$$

$$= 300\,\pi\ \mathrm{cm}^3/\mathrm{minute}$$ ✓ ✓✓

Question 6 (5 marks) CF

Given $v(x) = 9 + x^2$

$$\therefore \frac{dx}{dt} = 9 + x^2$$ ✓

Using separation of variables:

$$\frac{dx}{9 + x^2} = dt$$ ✓

$$\int \frac{1}{3^2 + x^2}\,dx = \int 1\,dt$$ ✓

$$\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) = t + c$$ ✓✓

iven $x = -3$ when $t = 0$

$$\frac{1}{3}\tan^{-1}(-1) = 0 + c$$

$$c = \frac{1}{3} \times \left(-\frac{\pi}{4}\right)$$ ✓

$$c = \frac{-\pi}{12}$$ ✓

6.

justify procedures and decisions by explaining mathematical reasoning

solve problem by applying:

- the velocity relationship $v = \frac{dx}{dt}$ ✓
- separation of variables procedure for solving differential, first-order equation ✓✓
- a suitable antiderivative of integrand ✓✓

$\therefore\ \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) = t - \frac{\pi}{12}$ ✓

$\therefore\ x = 3\tan\left(3t - \frac{\pi}{4}\right)$

When $t = \frac{\pi}{12}$

$x = 3\tan\left(3\left(\frac{\pi}{12}\right) - \frac{\pi}{4}\right)$

$= 3\tan(0)$ ✓

$= 0$

The particle's position at $t = \frac{\pi}{12}$ is confirmed at the origin. ✓

solve problem by applying:

- boundary conditions to determine the constant of integration ✓✓
- substitution to find displacement as a function of time ✓

evaluate reasonableness of solution ✓✓

Question 7 (4, 5 marks) CF, CU

Let *R* and *P* represent motion associated with the rock and the particle:

7a.

justify procedures and decisions by explaining mathematical reasoning

solve problem by finding:

- v for rock and particle ✓✓
- x for rock and particle ✓✓

a. Assuming downwards is negative:

Consider the motion of the rock:

$a_R = -10$

$v_R = \int -10dt = -10t + C$

$v_R(0) = 0 = C$

$\therefore v_R = -10t$ ✓

$x_R = \int -10t\ dt = -5t^2 + C$

Assuming origin is at ground level:

$x_R(0) = 20 = C$

$\therefore x_R = -5t^2 + 20$.(1) ✓

Consider the motion of the particle:

$a_P = -10$

$v_P = \int -10\ dt = -10t + c$

$v_P(0) = 12 = c$

$\therefore v_P = -10t + 12$ ✓

$x_P = \int (-10t + 12)\ dt = -5t^2 + 12t + c$

$x_P(0) = 0 = c$

$\therefore x_P(t) = -5t^2 + 12t$ ✓

The rock and particle are at the same position when

solve problem by finding:

- the meeting time using given same position information ✓✓

$x_R(t) = x_p(t)$:

$-5t^2 + 20 = -5t^2 + 12t$ ✓

$t = \frac{5}{3}$ s ✓

Substituting into (1) to find the meeting position:

$x_R\left(\frac{5}{3}\right) = -5\left(\frac{5}{3}\right)^2 + 20$ ✓

$= -\frac{125}{9} + 20$

$= -\frac{125}{9} + \frac{180}{9}$

$= \frac{55}{9}$

$= 6\frac{1}{9}$ m

evaluate reasonableness of estimation by determining actual meeting position above the ground ✓✓

The estimation of 10 m was not reasonable. ✓

7b.

justify procedures and decisions by explaining mathematical reasoning, by:

- making connection that the position model of the rock is unchanged ✓
- recognising that the position of the rock must be 10 (m) ✓
- finding the corresponding time ✓

solve problem by:

- refining the position model for the particle ✓✓
- making connection with meeting time and position of the rock and particle ✓✓
- using the common meeting time to find the initial velocity of the particle ✓ as an exact value ✓✓

b. Need to find the launch speed of the particle so that it meets the rock at 10 m above the ground.

$\therefore x_R = x_P = 10$

The position model of the rock is still effective: ✓

$\therefore x_R = -5t^2 + 20 = 10$ ✓

$-10 = -5t^2$

$\therefore t = \sqrt{2}$ ✓ (t > 0)

So, the rock and particle would meet at $t = \sqrt{2}\,s$.

Refining the position model of the particle:

New meeting height will be $x_P = -5t^2 + v_0 t$ where v_0 is the new launch speed of the particle. ✓✓

Require $x_P = 10$

$\therefore 10 = -5\left(\sqrt{2}\right)^2 + v_0 \times \left(\sqrt{2}\right)$ ✓✓

$\therefore -10 + \sqrt{2}v_0 = 10$

$\sqrt{2}v_0 = 20$ ✓

$\therefore v_0 = \frac{20}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}$ ✓✓

The particle must be launched at $10\sqrt{2}\,\mathrm{m\,s^{-1}}$.

8.

recall and use

- the probability distribution function by stating the appropriate integral using the correct lower and upper limits ✓✓✓✓
- technology to find the required probability ✓✓

communicate the result as required ✓✓

9a.

select and use rules to calculate both sample mean parameters ✓✓✓

9b.

communicate an understanding of the central limit theorem using everyday language ✓

recall procedure to find:

- the required probability using technology (or otherwise) ✓✓✓✓
- the number expected in 50 samples ✓✓

communicate the result as an integer value ✓✓

Paper 2 (technology-active)

Question 8 (4 marks) SF

$P(t < 10) = \int_0^{10} 0.04e^{-0.04t} dt$ ✓✓✓✓

$= 0.32968$ ✓✓ (using GDC)

There is a 33% chance. ✓✓

Question 9 (1.5, 4.5 marks) SF

a. $\mu_{\bar{X}} = \mu = 280\ 000$ ✓

$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{80\ 000}{\sqrt{200}} = 5656.85$ ✓✓

b. Since $n > 30$, an approximation to the normal distribution can be used. ✓

Using GDC (with the above statistics):

$P(\bar{X} > 290\ 000) = 0.0386$ ✓✓✓✓

Expected number $= np$

$= 50 \times 0.03856$

$= 1.928$ ✓✓

≈ 2 ✓✓

Question 10 (4 marks) SF

Since $n > 30$, an approximation to the normal distribution can be used. ✓

Given a normal distribution with $\bar{x} = 1280,\ s = 125.$ ✓

Confidence interval is $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$

Using GDC for $n = 100$ to find a 99% confidence interval ⇒ $(\$1247.80,\ \$1312.20)$ ✓✓✓✓

Since the suggested mean $\mu = \$1200$ lies outside of the 99% confidence interval, the union's claim is not valid based on this sample data. ✓✓

10.

communicate an understanding of the central limit theorem using everyday language ✓

recognise and select both sample mean parameters ✓

use an appropriate procedure to determine the required confidence interval ✓✓✓✓

evaluate the reasonableness of the claim ✓✓

11. recall and use the following procedures related to Simpson's rule: • find strip width ✓ • find suitable x values ✓ • use technology to calculate corresponding $f(x)$ values ✓✓ • recall Simpson's rule ✓ • substitute into rule correctly ✓✓ • calculate the approximate area ✓	**Question 11 (4 marks) SF** Given $f(x) = \frac{\ln(x+0.1)}{2.1} - x + 7.1$ Using 4 strips for $0 \leq x \leq 8 \Rightarrow w = 2$ ✓ $x_0 = 0,\ x_1 = 2,\ x_2 = 4, x_3 = 6, x_4 = 8$ ✓ $f(0) \approx 6.0035$ $f(2) \approx 5.4533$ $f(4) \approx 3.7719$ $f(6) \approx 1.9611$ $f(8) \approx 0.0961$ ✓✓ Area $\approx \frac{w}{3}[f(x_0) + f(x_n) + 4[f(x_1) + \cdots + f(x_{n-1})] + 2[f(x_2) + \cdots + f(x_{n-2})]\,]$ ✓ $\approx \frac{2}{3}\left[6.0035 + 0.0961 + 4[5.4533 + 1.9611] + 2[3.7719]\right]$ ✓✓ ≈ 28.8673 ✓ (m^2)	
12. justify procedures by explaining mathematical reasoning solve problem by: • establishing the appropriate statistical parameters ✓ • defining the required probability statement ✓ • using technology to determine an equivalent z_1-score ✓✓ • substituting correctly into a relevant probability equation ✓✓ • solving the equation to find the number of passengers ✓ communicate the result as an integer value ✓	**Question 12 (4 marks) CF** $\mu_{\bar{X}} = \mu = 17.3$ $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.1}{\sqrt{n}}$ ✓ $\bar{X}$ = mean luggage mass for a random sample of size n $P(\bar{X} < 16) = 0.0179$ ✓ For this probability, $z_1 = -2.097$ (using GDC) ✓✓ $z_1 = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ $\therefore\ -2.097 = \frac{16 - 17.3}{\frac{3.1}{\sqrt{n}}}$ ✓✓ $\frac{3.1}{\sqrt{n}} = \frac{16 - 17.3}{-2.097}$ $n = \left(\frac{3.1}{0.62}\right)^2 = 25.005$ ✓ There are 25 passengers chosen. ✓	

Question 13 (8 marks) CU

Gradient $= \frac{dy}{dx} = \frac{y^2 - 11y + 30}{\cos^2(x)}$ ✓

Using separation of variables:

$$\int \frac{dy}{y^2 - 11y + 30} = \int \frac{1}{\cos^2(x)}\, dx \checkmark$$

$$\int \frac{dy}{(y-5)(y-6)} = \int \sec^2(x)\ dx \text{ ... (A)}$$

Using partial fractions procedure:

$$\frac{1}{(y-5)(y-6)} = \frac{A}{y-5} + \frac{B}{y-6} \checkmark$$

$$1 = A(y-6) + B(y-5)$$

$$1 = Ay - 6A + By - 5B \checkmark$$

$$1 = (A+B)y + (-6A - 5B)$$

Equating coefficients:

$A + B = 0$ … (1)

$-6A - 5B = 1$ … (2) ✓

Substituting $A = -B$ into (2)

$$-6(-B) - 5B = 1$$

$$B = 1$$

$$\therefore A = -1 \quad \checkmark$$

Substituting into (A)

$$\therefore \int \sec^2(x)\, dx = \int \frac{-1}{y-5} + \frac{1}{y-6}\, dy$$

$$\tan(x) = \ln\left(\frac{y-6}{y-5}\right) + c \quad \checkmark\checkmark$$

$$\text{iven} \quad x = 0,\ y = 7$$

$$-c = \ln\left(\frac{1}{2}\right)$$

$$\therefore\ c = \ln 2 \checkmark$$

Finding the required function:

$$\tan(x) = \ln\left(\frac{y-6}{y-5}\right) + \ln 2$$

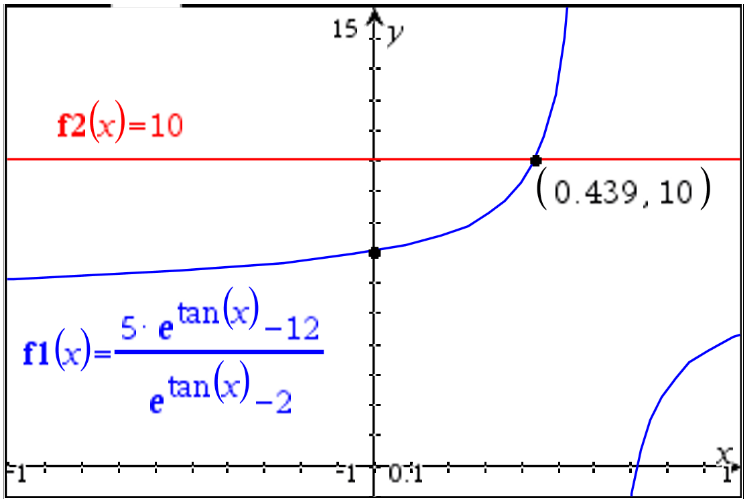
$$\tan(x) = \ln\left(\frac{2(y-6)}{y-5}\right) \checkmark$$

$$e^{\tan(x)} = \frac{2y - 12}{y - 5}$$

13.

justify procedures by explaining mathematical reasoning, by:

- forming a differential equation of the form $\frac{dy}{dx} = f(x)g(y)$ ✓
- recognising the use of separation of variables procedure for solving the differential equation ✓
- generating integral functions using partial fractions ✓✓✓✓
- using suitable integration methods to find a general solution to the differential equation ✓✓
- applying the boundary condition to determine the constant of integration ✓
- substituting constant and using algebraic rearrangement to express y as a function of x ✓✓✓

	$e^{\tan(x)}(y-5) = 2y - 12$ $ye^{\tan(x)} - 5e^{\tan(x)} = 2y - 12$ $ye^{\tan(x)} - 2y = 5e^{\tan(x)} - 12$ ✓ $y(e^{\tan(x)} - 2) = 5e^{\tan(x)} - 12$ $y = \frac{5e^{\tan(x)} - 12}{e^{\tan(x)} - 2}$ ✓ Using a graphical method to find the smallest positive value of x when $y = 10$. ✓✓  The intersection point shows $x \approx 0.439$. ✓✓	use suitable technology to: • determine possible solutions for x ✓✓ • ensure that the solution selected is the smallest positive value ✓✓