Specialist Mathematics 2019 v1.2

Unit 2 sample marking scheme

October 2018

Examination

This sample has been compiled by the QCAA to model one possible approach to allocating marks in an examination. It matches the examination mark allocations as specified in the syllabus (~ 60% simple familiar, ~ 20% complex familiar and ~ 20% complex unfamiliar) and ensures that all assessment objectives are assessed.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 2 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 2 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 2 topics.

Task

See the sample assessment instrument for Unit 2: Examination (available on the QCAA Portal).

Sample marking scheme

The annotations are written descriptions of the expected response for each question and are related to the assessment objectives.

Note: ✓ = $\frac{1}{2}$ mark

Paper 1 (technology-free)

Question 1 (1 mark) SF

1.
recall and use:
- the definition of i^2 ✓
- the procedure to result ✓

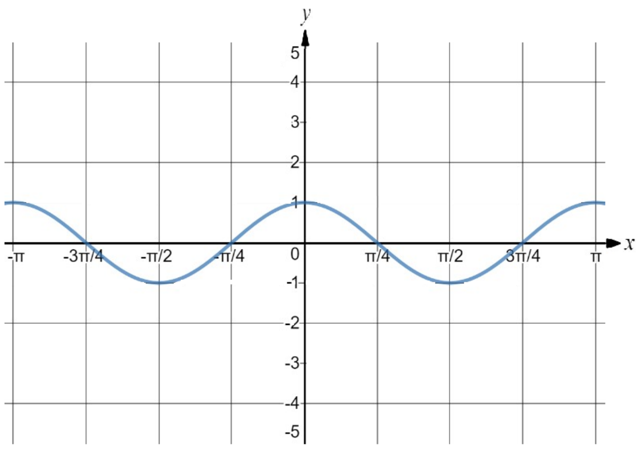
$$3i^2 + 2 = 3(-1) + 2 \checkmark$$
$$= -1 \checkmark$$

Question 2 (6 marks) SF

2a.

sketch $y = \cos(2x)$ by identifying the following three features:
- general shape ✓
- correct position of the maximum and minimum values ✓
- correct position of the zeros of the function ✓

a. ✓✓✓



b. The reciprocal function of $y = \sec(2x)$ is $y = \cos(2x)$. So $y = \sec(2x)$ can be sketched based on the sketch of $y = \cos(2x)$ by applying the general graphical properties linking reciprocal functions. ✓✓

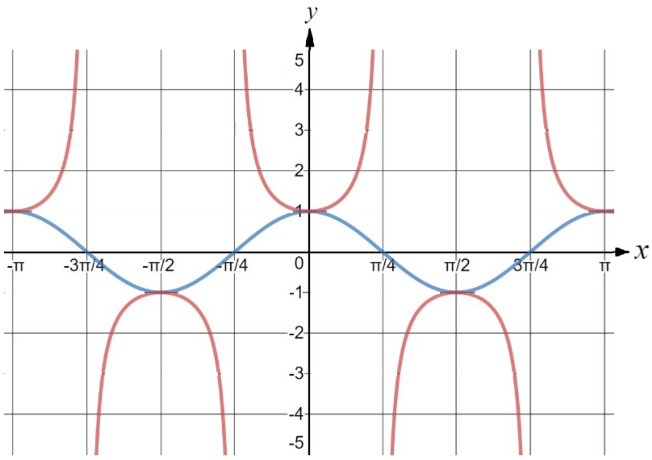
2b.
comprehend the reciprocal function relationship of $y = \cos(2x)$ and $y = \sec(2x)$ ✓
communicate that reciprocal functions have graphical links ✓

2c.
sketch $y = \sec(2x)$ by identifying the following six features:

- general shape ✓
- $f(x)$ and $\frac{1}{f(x)}$ have the same sign for any respective domain ✓
- as $f(x)$ increases, $\frac{1}{f(x)}$ decreases and vice versa ✓
- the zeros of $y = f(x)$ correspond to the asymptotes of $y = \frac{1}{f(x)}$ ✓
- the asymptotes of $y = f(x)$ correspond to the zeros of $y = \frac{1}{f(x)}$ ✓
- maximum values of $y = f(x)$ correspond to minimum values of $y = \frac{1}{f(x)}$ and vice versa ✓

communicate by clearly identifying each function ✓

c. ✓✓✓✓✓✓



Key (*or other suitable method of labelling*):

$y = \cos(2x)$ — blue

$y = \sec(2x)$ — red ✓

Question 3 (2 marks) SF

$$
\begin{aligned}
|z| &= \sqrt{(-\sqrt{5})^2 + (7)^2} \quad ✓ \\
&= \sqrt{5 + 49} \quad ✓ \\
&= \sqrt{54} \quad ✓ \\
&= 3\sqrt{6} \quad ✓
\end{aligned}
$$

3.
recall and use:

- a suitable rule to determine the modulus ✓
- the rule $(\sqrt{a})^2 = a$ ✓
- a suitable procedure to a single surd ✓
- the exact result in simplified form ✓

Question 4 (6 marks) SF

4a.
recall definition and use of a translation ✓✓

4b.
recall:

- the application of a linear transformation using a 2×2 matrix ✓✓
- the process to determine image vertices ✓✓

communicate by clearly identifying the image vertices L, M, N ✓

a. The vector of translation is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. ✓✓

b. The images of D(0, –2), E(–1, –1), F(–2, –3) are given by

$$\begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & -2 \\ -2 & -1 & -3 \end{bmatrix} \quad ✓✓$$

$$= \begin{bmatrix} 2 & 3 & 7 \\ -2 & 2 & 3 \end{bmatrix} \quad ✓✓$$

The image triangle has vertices at L(2, –2), M(3, 2) and N(7, 3) ✓

c. The area of ΔABC is equal to the area of ΔDEF as translations preserve the area of the original shape.✓

$$
\begin{aligned}
\frac{\text{Area of }\Delta\text{LMN}}{\text{Area of }\Delta\text{DEF}} &= |\det(T)| \quad ✓✓ \\
&= |-2 \times 1 - \ -3 \times -1| \\
&= |-5| \quad ✓ \\
&= 5
\end{aligned}
$$

So ΔLMN is 5 times larger than ΔABC. ✓

4c.
describe and/or determine the relationship between the areas of triangles ABC and DEF ✓

comprehend the link between determinant and area factors of original and image triangles ✓✓

use the process to calculate the relationship between the areas of triangles DEF and LMN ✓

communicate the required factor as a positive value ✓

Question 5 (3 marks) SF

RTP: $\cos(3\theta)\cos(\theta) = \cos^2(\theta) - \sin^2(2\theta)$

LHS $=\cos(3\theta)\cos(\theta)$

$= \frac{1}{2}(\cos(2\theta) + \cos(4\theta))$ ✓✓

$= \frac{1}{2}(2\cos^2(\theta) - 1 + 1 - 2\sin^2(2\theta))$ ✓ ✓✓

$= \cos^2(\theta) - \sin^2(2\theta)$ ✓

= RHS

5.
select and use:

- the required product of cosines into a sum rule ✓✓
- the required double angle identity rule for cos (2θ) ✓
- the required double angle identity rule for cos (4θ) ✓✓
- suitable procedure to complete proof ✓

Question 6 (4 marks) CF

RTP

$|\mathbf{A}| = \frac{1}{|\mathbf{A}^{-1}|}$

LHS = $|\mathbf{A}|$

$= ad - bc$ ✓

RHS = $\frac{1}{|\mathbf{A}^{-1}|}$

$= \frac{1}{\begin{vmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{vmatrix}}$ ✓ (given $\mathbf{A}$ is non-singular)

$= \frac{1}{\frac{d}{ad-bc}\frac{a}{ad-bc} - \frac{-b}{ad-bc}\frac{-c}{ad-bc}}$ ✓

$= \frac{1}{\frac{ad-bc}{(ad-bc)^2}}$ ✓✓

$= ad - bc$ ✓

= LHS

Hence $|\mathbf{A}| = \frac{1}{|\mathbf{A}^{-1}|}$ is proven as required. ✓✓

6.
recognise the notation of $|\mathbf{A}|$ ✓

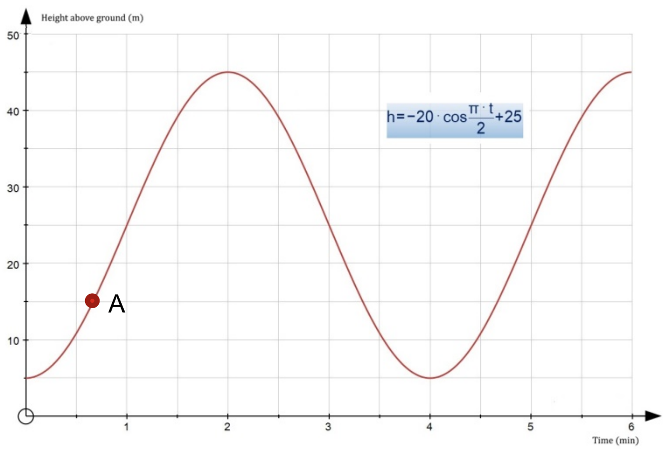
select and use the rule for the determinant of $\mathbf{A}^{-1}$ ✓✓

prove the proposition by explaining mathematical reasoning ✓✓✓

communicate using mathematical organisation of proof conventions ✓✓

Question 7 (4, 5 marks) SF, CF

a. ✓✓✓✓



7a.
sketch by identifying the following four features on the graph:

- appropriate sinusoidal shape ✓
- period of 4 minutes ✓
- minimums of 5 metres at $t = 0$ ✓
- maximums of 45 metres at $t = 2$ and $t = 6$ minutes ✓

7b.
recall rules and justify procedures to find:

- amplitude ✓
- angular frequency ✓✓
- vertical shift ✓

b. Amplitude = $a = \frac{45-5}{2}$ = 20 m ✓

Angular frequency = $b = \frac{2\pi}{period}$ ✓

$= \frac{2\pi}{4} = \frac{\pi}{2}$ ✓

Vertical shift = d = 25 m ✓

c. Since the position of the passenger is assumed to start at a minimum value, and is periodic in nature, a negative cosine function is an appropriate model.

$h = -a\cos(bt) + d$ ✓✓✓

where h = height above ground (metres)

t = time (minutes) ✓

$\therefore\ h = -20\cos\left(\frac{\pi t}{2}\right) + 25$ ✓

When t = 40 seconds = $\frac{2}{3}$ minute ✓

$$h = -20\cos\left(\frac{\pi \times \frac{2}{3}}{2}\right) + 25$$

$$= -20\cos\left(\frac{\pi}{3}\right) + 25 \quad ✓$$

$$= -20 \times \frac{1}{2} + 25$$

$$= 15\text{ m} \quad ✓$$

7c.
solve the problem through the synthesis of an appropriate model ✓✓✓

communicate:

- by defining variables (including units) ✓
- by stating a suitable model using the determined parameters ✓

recall and use procedure to:

- calculate in the defined units ✓
- calculate using exact trigonometric values ✓
- simplified accurate result ✓

d. As indicated on the sketch (see point A), at $t = \frac{2}{3}$ minutes, the value of h is close to 15 m so this result is reasonable. ✓✓

7d.
evaluate the reasonableness of solution ✓✓

Question 8 (7 marks) CU

8a.
solve the problem by analysing and synthesising the effect of the product of complex numbers and the transformations:

- dilation ✓✓
- rotation ✓✓
- combined effect ✓

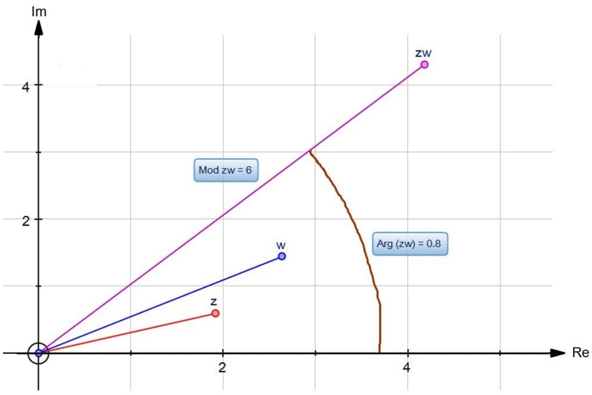
(The use of an Argand diagram is an ***optional*** method of comprehending the transformations associated with multiplying a complex number by w.)

a. Multiplying any complex number by w increases the modulus of the original number by $|w|$ which represents a dilation factor of 3. ✓✓

Multiplying any complex number by w increases the argument of the original number by $\arg(w)$ which represents a rotation about the origin of 0.5. ✓✓

So, the act of multiplying a complex number by w has the combined effect of a dilatation of factor of 3 and a rotation about the origin of 0.5 radians. ✓

(Alternatively, this understanding can be demonstrated by representing the product of w and another complex number, say $z = 2cis(0.3)$, on an Argand diagram as shown below).



Note: The order of the transformations is not significant, because the matrix representing a dilation is a scalar multiple of the identity matrix, which is commutative under multiplication.

select and use rule for the standard matrix transformation of: • dilation ✓ • rotation ✓ justify procedure to determine a matrix representing a single transformation ✓	This complex number product corresponds to the following two transformations, written in matrix form: $\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ✓ $\mathbf{R} = \begin{bmatrix} \cos(0.5) & -\sin(0.5) \\ \sin(0.5) & \cos(0.5) \end{bmatrix}$ ✓ Forming a single matrix of transformation: $\mathbf{T} = \mathbf{DR} = \begin{bmatrix} 3\cos(0.5) & -3\sin(0.5) \\ 3\sin(0.5) & 3\cos(0.5) \end{bmatrix}$ ✓	
8b. justify the result using conceptual understanding to connect information through: • translation of mathematics using the isomorphism between complex numbers and matrices ✓✓✓ • application of a suitable addition trigonometric identity ✓✓ evaluate the reasonableness of solution ✓	b. Evaluating the reasonableness of the solution. $wz = 3\text{cis}(0.5)2\text{cis}(0.3)$ $= 6\ \text{cis}(0.3 + 0.5)$ $= 6\text{cis}(0.8)$ ✓ Expressing z as a column matrix $\begin{bmatrix} 2\cos(0.3) \\ 2\sin(0.3) \end{bmatrix}$ ✓ Applying $\mathbf{T}$, wz corresponds to $\begin{bmatrix} 3\cos(0.5) & -3\sin(0.5) \\ 3\sin(0.5) & 3\cos(0.5) \end{bmatrix} \begin{bmatrix} 2\cos(0.3) \\ 2\sin(0.3) \end{bmatrix}$ $= \begin{bmatrix} 6\cos(0.5)\cos(0.3) - 6\sin(0.5)\sin(0.3) \\ 6\sin(0.5)\cos(0.3) + 6\cos(0.5)\sin(0.3) \end{bmatrix}$ $= \begin{bmatrix} 6[\cos(0.5)\cos(0.3) - \sin(0.5)\sin(0.3)] \\ 6[\sin(0.5)\cos(0.3) + \cos(0.5)\sin(0.3)] \end{bmatrix}$ ✓ Applying the addition trigonometrical identity: $\cos(A)\cos(B) - \sin(A)\sin(B) = \cos(A + B)$ and $\sin(A)\cos(B) + \cos(A)\sin(B) = \sin(A + B)$ $\therefore wz$ is equivalent to $\begin{bmatrix} 6\cos(0.8) \\ 6\sin(0.8) \end{bmatrix}$ ✓✓ Expressing this result back as a complex number, $wz = 6\text{cis}(0.8)$. So, the matrix of transformation $\mathbf{T}$ is shown to be a reasonable solution. ✓	
	Paper 2 (technology-active)	
9a. use technology: • to find the $arg(w)$ ✓✓ (alternatively, an analytic process could have been used) • using degree mode ✓ communicate result using correct rounding ✓	**Question 9 (4 marks) SF** a. $arg(w)$ = –63.43° (or 296.57°) ✓✓✓ ≈ –63° (or 297°) ✓ b. $Im(w^6) = Im((1 - 2i)^6)$ $= Im(117 - 44i)$ ✓✓ $= -44$ ✓✓	**9b.** use technology to find w^6 ✓✓ (alternatively, an algebraic method could have been used) recall and use of the definition of the imaginary component ✓✓

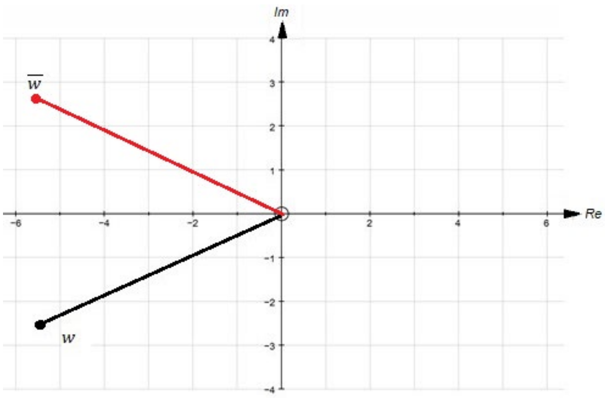
10a.
recall the definition of conjugate ✓
communicate the position of $\overline{w}$ clearly on the diagram ✓

10b.
recall and use:
- cis form ✓
- a suitable conversion method to Cartesian form ✓✓✓

communicate the result using correct rounding
evaluate the reasonableness of the result ✓✓

Question 10 (4 marks) SF

a. ✓✓



b. $w = 6.0\,cis(-147.5°)$ ✓

$= 6.0\cos(-147.5) + 6.0\sin(-147.5)\,i$ ✓✓

$= -5.06 - 3.22i$ ✓

The position of w on the Argand diagram in the 3rd quadrant is reasonable but the Cartesian coordinates do not match closely. There must be an error with the estimation. ✓✓

11.
select the suitable Pythagorean identity ✓
recall and use procedures to:
- find $cot^2\alpha$ ✓
- determine two solutions for $cot\,\alpha$ ✓✓
- solve the equation by applying the given domain ✓✓

Question 11 (3 marks) SF

Given $\text{cosec}(\alpha) = 1.25$

Using a Pythagorean identity,

$\text{cosec}^2\alpha = 1 + \cot^2\alpha$ ✓

$1.25^2 = 1 + \cot^2\alpha$

$1.25^2 - 1 = \cot^2\alpha$

$0.5625 = \cot^2\alpha$ ✓

$\cot\alpha = \pm 0.75$ ✓✓

Given $\frac{\pi}{2} \le \alpha \le \pi$, $\cot\alpha = -0.75$ ✓✓

12a.
recognise and use matrix notation ✓✓

12b.
use technology to perform the following matrix calculations:
- scalar product ✓
- power of a matrix ✓✓
- matrix subtraction ✓

Question 12 (3 marks) SF

a. $b_{2,1} = -4$ ✓✓

b. $2\mathbf{A} - \mathbf{B}^2 = \begin{bmatrix} -4 & -6 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} -16 & 10 \\ -8 & -20 \end{bmatrix}$ ✓ ✓✓

$= \begin{bmatrix} 12 & -16 \\ 14 & 24 \end{bmatrix}$ ✓

13.
recall the procedure to solve a matrix equation:
- recognise use of an inverse matrix ✓
- recognise use of post-multiplication ✓
- correctly establish $\mathbf{X}$ as the subject of the equation using given matrices ✓✓
- use technology to solve equation ✓✓✓✓

Question 13 (4 marks) SF

Given $\mathbf{XA} = \mathbf{B}$

$\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$ ✓

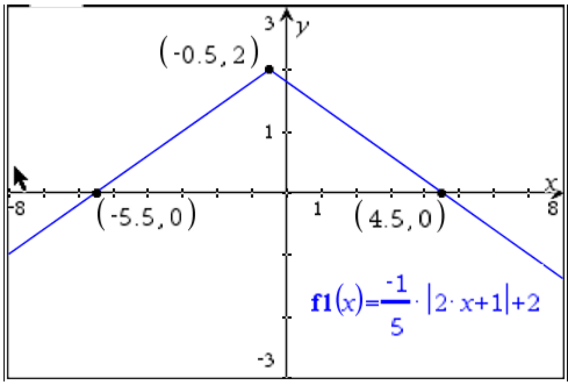
$\mathbf{X} = \mathbf{BA}^{-1}$ ✓

$= [-2.2 \quad 10.4 \quad 9.3]\begin{bmatrix} -1 & 2 & 4 \\ -2 & 0 & 3 \\ 1 & -4 & -3 \end{bmatrix}^{-1}$ ✓✓

$= [1.2 \quad -0.5 \quad -2]$ ✓✓✓✓

Question 14 (3 marks) SF

✓✓✓✓✓✓



14.
identify the following six features of the function on the sketch using technology:

- appropriate absolute value shape ✓
- use (full) domain of $-8 \leq x \leq 8$ ✓
- labels to indicate the position of the vertex and x-intercepts ✓
- vertex at (–0.5, 2) ✓
- x-intercept at (–5.5, 0) ✓
- x-intercept at (4.5, 0) ✓

Question 15 (6 marks) CF

15a.
solve the problem by synthesising the following:

- the conjugate root theorem ✓✓
- the factor theorem ✓✓

use of algebraic procedures related to the quadratic function ✓✓✓✓✓

a. Given that $z = 2 - 3i$ is a root of the real quadratic equation, $\bar{z} = 2 + 3i$ is also a root. ✓✓

$\therefore f(z) = (z - (2 - 3i))(z - (2 + 3i))$ ✓✓

Expanding:

$f(z) = z^2 - (2 + 3i)z - (2 - 3i)z + (2 - 3i)(2 + 3i)$ ✓

$= z^2 - 2z - 2z - 3iz + 3iz + 4 - 9i^2$ ✓✓✓

$\therefore f(z) = z^2 - 4z + 13$ ✓

b. Verifying by showing $f(2 - 3i) = 0$ ✓

LHS $= (2 - 3i)^2 - 4(2 - 3i) + 13$

= 0

= RHS ✓✓

The result is reasonable.

15b.
recognise a suitable strategy to verify the solution ✓

use technology (or otherwise) to evaluate the reasonableness of the solution ✓✓

(an alternative verification could have been completed by showing $f(2 + 3i) = 0$)

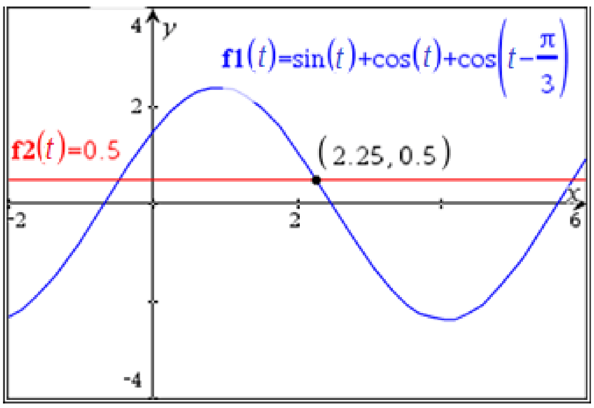
Question 16 (2, 8 marks) SF, CU

16a.
justify the solution using technology to sketch the:

- given function ✓
- line $y = 0.5$ ✓
- required intersection point ✓

a. Finding the first intersection after $t \geq 0$ for the graphs of $x = \sin(t) + \cos(t) + \cos\left(t - \frac{\pi}{3}\right)$ and $x = 0.5$.

✓✓✓



The initial time that the particle is positioned 0.5 centimetres from the central point is after approximately 2.25 seconds. ✓

communicate the required time of ≈ 2.25 seconds ✓

16b. use conceptual knowledge to identify the need to expand the expression $\cos\left(t-\frac{\pi}{3}\right)$ using the identity $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$) ✓	b. $x = \sin(t) + \cos(t) + \cos\left(t-\frac{\pi}{3}\right)$ for $t \geq 0$. $x = \sin(t) + \cos(t) + \cos(t)\, cos\left(\frac{\pi}{3}\right) + \sin(t)\, sin\left(\frac{\pi}{3}\right)$ ✓ $= \sin(t) + \cos(t) + \frac{1}{2}\cos(t) + \frac{\sqrt{3}}{2}\sin(t)$ ✓ $= \left(\frac{\sqrt{3}+2}{2}\right)\sin(t) + \frac{3}{2}\cos(t)$ ✓✓	
recall and use: • exact trigonometric values ✓ • simplification of trigonometric expressions ✓✓	Equating to the form $x = r\cos(t-\alpha)$ gives ✓ $x = \left(\frac{\sqrt{3}+2}{2}\right)\sin(t) + \frac{3}{2}\cos(t)$ $= r\cos(t)\cos(\alpha) + r\sin(t)\sin(\alpha)$ ✓	
justify procedures using the following mathematical reasoning: • modelling the displacement in terms of a single trigonometric ratio ✓ • expressing the sum of two trigonometric terms as a single term using a suitable trigonometric identity ✓	Equating coefficients and applying trigonometrical identities: $\frac{3}{2} = r\cos(\alpha)$ *and* $\frac{\sqrt{3}+2}{2} = r\sin(\alpha)$ ✓ $r^2(sin^2(\alpha) + cos^2(\alpha)) = \left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}+2}{2}\right)^2$ ✓ $r^2 = 4 + \sqrt{3}$ $r = \sqrt{4+\sqrt{3}} \approx 2.394$ ✓ $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$ ✓ $= \frac{\frac{(\sqrt{3}+2)}{2}}{\frac{3}{2}}$ $= \frac{(\sqrt{3}+2)}{3}$ ✓	
use procedures to determine both parameters ✓✓✓✓✓✓	$\alpha = \tan^{-1}\left(\frac{\sqrt{3}+2}{3}\right) \approx 0.894$ ✓	
communicate the refined model ✓	The particle can be modelled by $x = 2.394\cos(t - 0.894)$ for $t \geq 0$ ✓	
use algebraic method to evaluate the reasonableness of solution ✓✓✓	When $x = 0.5$, $0.5 = 2.394\cos(t - 0.894)$ ✓ $t - 0.894 = \cos^{-1}\left(\frac{0.5}{2.394}\right)$ ✓ $t \approx 1.360 + 0.894$ ≈ 2.25 s ✓ So, the reasonableness of the solution from part (a) has been verified.	