Specialist Mathematics 2019 v1.2

Unit 1 sample marking scheme

June 2019

Examination

This sample has been compiled by the QCAA to model one possible approach to allocating marks in an examination. It matches the examination mark allocations as specified in the syllabus (~ 60% simple familiar, ~ 20% complex familiar and ~ 20% complex unfamiliar) and ensures that all assessment objectives are assessed.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 1 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 1 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 1 topics.

Task

See the sample assessment instrument for Unit 1 Topics 1–3: Examination (available on the QCAA Portal).

Sample marking scheme

Criterion	Marks allocated	Result
Foundational knowledge and problem-solving Assessment objectives numbers 1, 2, 3, 4, 5, 6		

The annotations are written descriptions of the expected response for each question and are related to the assessment objectives.

Note: ✓ = $\frac{1}{2}$ mark

Part A — Simple familiar

Question 1 (2 marks)

Number of ways = 24 × 23 × 22 ✓ ✓✓
= 12 144 ✓

1.

comprehend the connection with the multiplication principle or permutation $^{24}P_3$ ✓

use of:

- relevant values ✓✓
- a suitable procedure to accurate result ✓

Question 2 (1½, 4½ marks)

(a) Number of committees = $^{15}C_5$ ✓✓
= 3003 ✓

2a.

comprehend the connection with a combination ✓

use of:

- relevant values ✓
- technology or procedure to accurate result ✓

(b) Number of committees with all captains = $^{3}C_3 \times {}^{12}C_2$ ✓ ✓
= 66

Number of committees with two captains = $^{3}C_2 \times {}^{12}C_3$ ✓ ✓
= 660

Number with at least two captains = 66 + 660 ✓✓
= 726

P(committee includes at least two captains) = $\frac{726}{3003}$ ✓
= $\frac{22}{91}$ (≈ 0.242) ✓✓

2b.

comprehend the connection with

- the multiplication principle ✓✓
- the addition principle ✓✓

use of:

- relevant values ✓✓
- suitable procedures to probability result (allow carried error) ✓

communicate clearly through each step using correct mathematical notation and everyday language ✓✓

Question 3 (3 marks)

Number of doors:
Consider the 2 red doors together as a single 'door' ✓
Green: 3; Blue: 1; Orange: 4
Total 'doors' to be arranged = 9 ✓
Number of arrangements with red doors together = $\frac{n!}{n_1!n_2!n_3!}$ ✓
= $\frac{9!}{3!1!4!}$ ✓✓
= 2520 ✓

3.

communicate that the red doors are to be considered 'as a single unit' ✓

comprehend the need to use permutations with identical objects ✓

use of:

- relevant values ✓
- a rule related to permutations with identical objects ✓✓
- technology to calculate the result ✓

Question 4 (3 marks)

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ ✓✓

$= \begin{pmatrix}-7\\2\end{pmatrix} - \begin{pmatrix}-3\\-1\end{pmatrix}$ ✓

$= \begin{pmatrix}-4\\3\end{pmatrix}$ ✓

In polar form, $\overrightarrow{AB}$ = [5, 143.13°] ✓✓

4.

recall:

- the use a suitable displacement vector rule ✓✓
- the use a suitable procedure to determine the result ✓✓
- the use of technology or analytic procedure to convert to polar form (allow carried error) ✓✓

Question 5 (3½, 2½ marks)

a. $\boldsymbol{a}.\boldsymbol{b} = |\,\boldsymbol{a}\,|\,|\,\boldsymbol{b}\,|\cos\theta$ ✓

$\begin{pmatrix}2\\6\end{pmatrix}.\begin{pmatrix}7\\-1\end{pmatrix} = \sqrt{2^2+6^2}\,\sqrt{7^2+(-1)^2}\cos\theta$

$\cos\theta = \frac{14-6}{\sqrt{40}\sqrt{50}}$ ✓✓✓

$\theta = \cos^{-1}\frac{8}{\sqrt{2000}}$

$= 79.69\ldots$ ✓✓

$\theta \approx 80°$ ✓

b. Projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ = $(\boldsymbol{a}\,.\,\hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$

$= \begin{pmatrix}2\\6\end{pmatrix}.\frac{1}{\sqrt{50}}\begin{pmatrix}7\\-1\end{pmatrix}\hat{\boldsymbol{b}}$ ✓

$= \frac{8}{\sqrt{50}}\hat{\boldsymbol{b}}$ ✓✓

$= \frac{4}{25}\begin{pmatrix}7\\-1\end{pmatrix}$ ✓✓

5a.

select and use:

- the scalar product definition ✓
- a rule to calculate the scalar product of two vectors ✓✓
- a rule to determine the magnitude of vectors ✓

recall and use a suitable procedure to determine the angle (allow carried error) rounded to the nearest degree ✓✓

communicate in organised form using mathematical conventions ✓

5b.

select and use:

- the definition of vector projection ✓
- a suitable procedure to determine the result (allow carried error) ✓✓

communicate the result in exact form using a rational denominator ✓✓

Question 6 (2 marks)

$\overrightarrow{DM} = \overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB}$ ✓ ✓

$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{b})$✓

$= \boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$ ✓

6.

use an appropriate procedure recognising the following concepts:

- (triangle) sum of vectors ✓
- midpoint ✓
- equivalence of vectors ✓

communicate the result using suitable vector notation ✓

Question 7 (1½, 3½ marks)

a. $3^2 = 9$ ✓

Since 9 is an odd number, this counter-example proves that the statement is false. ✓✓

b. RTP: 'The square of any odd number is always another odd number'

Odd numbers have the form $2n-1$ where $n \in N$ ✓✓
The square of an odd number is

$(2n-1)^2 = 4n^2 - 4n + 1$ ✓

$= 2(2n^2 + 2n + 1) - 1$ ✓✓

$= 2p - 1$ where $p \in N$ ✓✓

So, the statement is proven.

7a.

communicate:

- an appropriate counter-example ✓
- by stating a suitable description to explain why the statement is false ✓✓

7b.

communicate the form of an odd number using suitable mathematical notation ✓✓

recall and use a suitable procedure to:

- determine the expanded form of the square of an odd number ✓
- express the square back in the form of an odd number ✓✓
- prove the given statement ✓✓

Question 8 (2 marks)

a. 'If the scalar product of two vectors is not zero, then they are not perpendicular'. ✓✓

b. True ✓

Since the original statement is true, then the contraposition statement must be true. ✓

8a.

comprehend by articulating the contraposition statement ✓✓

8b.

comprehend by stating the correct decision ✓

communicate by giving a suitable brief explanation ✓

9.

recall and use a suitable procedure to complete the proof ✓✓✓✓✓

communicate using relevant:

- organisation of mathematical symbols/notation ✓
- everyday language ✓✓

Question 9 (4 marks)

$\Rightarrow \quad a = 2k,\ k \in Z$ …(continued)

$But \quad a^2 = 2b^2$ ✓

$\Rightarrow \quad (2k)^2 = 2b^2$ ✓

$\Rightarrow \quad 2b^2 = 4k^2$ ✓

$\Rightarrow \quad b^2 = 2k^2$ ✓

$\Rightarrow \quad b^2$ is even

$\Rightarrow \quad b$ is even ✓

✓

So, a and b are both even so they have a common factor of 2.

This contradicts the assumption that $\sqrt{2}$ is an irrational number. ✓✓

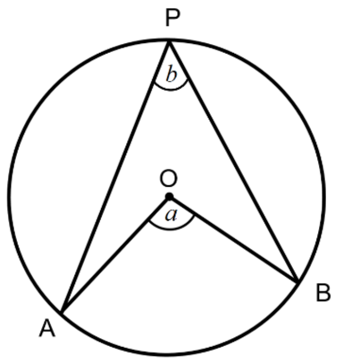
10a.

recall and use a suitable procedure:

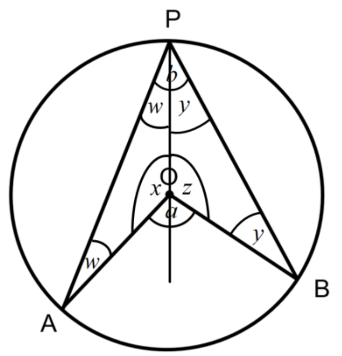
- recognising the need for a suitable construction ✓
- recognising the angle properties
  - of an isosceles triangle and its angle sum ✓
  - around a point ✓
- to accurately complete the proof ✓✓✓

Question 10 (3, 3 marks)

a. RTP: $a = 2b$.



Construct another radius to form two isosceles triangles with base angles w and y: ✓



Consider ∆OAP and ∆OBP

$x = 180° - 2w$ and $z = 180° - 2y$ ✓ … (1)

Consider the angles at O:

$a + x + z = 360°$ ✓

Substituting using (1):

$a + (180° - 2w) + (180° - 2y) = 360°$ ✓

$a - 2w - 2y = 0$

$a = 2(w + y)$ ✓

Since $b = w + y$, it is proven that $a = 2b$. ✓

10b.

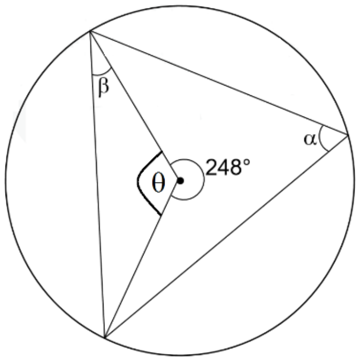
comprehend by identifying the value of:

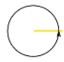
- angle at the centre ✓
- angle at the circumference (α) ✓
- base angle of the isosceles triangle ✓✓

communicate by:

- clearly describing the unknown angle at the centre ✓
- consistent use of relevant codes ✓

b. Adding angle θ to the diagram ✓



θ = 360° – 248°
= 112° ✓ Reason: 

$\alpha = \frac{112°}{2}$ Reason: 

= 56° ✓

$\beta = \frac{180° - 112°}{2}$ ✓ Reason: ✓ 

= 34° ✓

Part B — Complex familiar

Question 11 (6 marks)

Given ABCD is a quadrilateral

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 0$ ✓

$\therefore \overrightarrow{AB} + \overrightarrow{DA} = -\overrightarrow{CD} - \overrightarrow{BC}$ ✓ ... (1)

RTP: $\overrightarrow{MN} = \overrightarrow{QP}$ ✓✓

(as MN and QP must be parallel and equal in length)

LHS $= \overrightarrow{MN}$

$= \overrightarrow{MA} + \overrightarrow{AN}$

$= \frac{1}{2}\overrightarrow{DA} + \frac{1}{2}\overrightarrow{AB}$

$= \frac{1}{2}(\overrightarrow{DA} + \overrightarrow{AB})$ ✓✓

RHS $= \overrightarrow{QP}$

$= \overrightarrow{QC} + \overrightarrow{CP}$

$= \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{CB}$

$= \frac{1}{2}(\overrightarrow{DC} + \overrightarrow{CB})$ ✓✓

$= \frac{1}{2}(-\overrightarrow{CD} - \overrightarrow{BC})$ ✓

$= \frac{1}{2}(\overrightarrow{DA} + \overrightarrow{AB})$ … using (1) ✓

= LHS

It is proven that the midpoints of the sides of the given quadrilateral join to form a parallelogram. ✓✓

11.

use conceptual understanding to make connections between vectors and properties:

- of a quadrilateral ✓
- of a parallelogram ✓

translate information into vector form related to:

- displacements ✓✓
- midpoints ✓✓

prove proposition by:

- justifying procedures ✓✓
- explaining mathematical reasoning ✓✓

communicate using:

- relevant vector notation ✓
- an appropriate method of proof format ✓

12.

communicate by:

- organising and presenting information in graphical form ✓✓
- using mathematical terminology/symbols to represent mathematical models ✓
- defining variables ✓

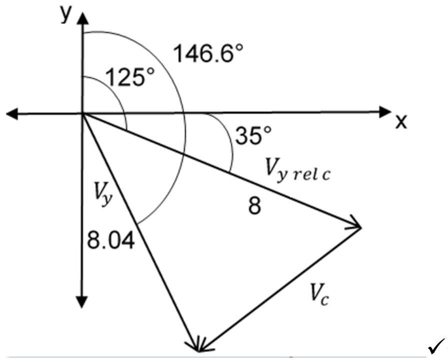
justify and explain processes:

- to calculate relative velocity ✓✓
- to convert vectors from polar to Cartesian form ✓✓
- to determine the velocity of the current ✓✓
- to determine the direction of the current (allow carried error/s) ✓✓

consider the reasonableness of solution by considering the original estimation ✓✓

Question 12 (7 marks)

Representing the situation using vectors:



✓✓

$\boldsymbol{V_y}$ — velocity of yacht (knots)

$\boldsymbol{V_c}$ — velocity of current (knots) ✓✓

$\boldsymbol{V_y} = 8.04\cos 303.4°\,\hat{\boldsymbol{\imath}} + 8.04\sin 303.4°\,\hat{\boldsymbol{\jmath}}$ ✓

$\approx 4.426\,\hat{\boldsymbol{\imath}} - 6.712\,\hat{\boldsymbol{\jmath}}$

$\boldsymbol{V_{y\,rel\,c}} = 8\cos 325°\,\hat{\boldsymbol{\imath}} + 8\sin 325°\,\hat{\boldsymbol{\jmath}}$ ✓

$\approx 6.553\,\hat{\boldsymbol{\imath}} - 4.589\,\hat{\boldsymbol{\jmath}}$

Determining velocity of the current:

$\boldsymbol{V_y} = \boldsymbol{V_{y\,rel\,c}} + \boldsymbol{V_c}$ ✓

$\therefore\ \boldsymbol{V_c} = \boldsymbol{V_y} - \boldsymbol{V_{y\,rel\,c}}$ ✓

$\boldsymbol{V_c} \approx (4.426\hat{\boldsymbol{\imath}} - 6.712\hat{\boldsymbol{\jmath}}) - (6.553\hat{\boldsymbol{\imath}} - 4.589\hat{\boldsymbol{\jmath}})$

$\approx -2.127\hat{\boldsymbol{\imath}} - 2.124\hat{\boldsymbol{\jmath}}$ ✓✓

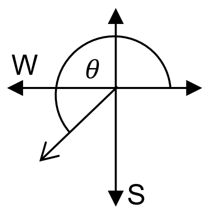
Determining direction in Quadrant 3:

$\tan\theta = \frac{opp}{adj}$

$\tan\theta = \frac{-2.214}{-2.217}$ ✓

$\theta = 44.96° + 180°$

$\approx 225°$ ✓



So, the direction of the current is running towards the south-west. The captain's estimation is reasonable. ✓✓

Part C — Complex unfamiliar

13.

analyse, generalise and translate the information into a mathematically workable format using combinatorial notation ✓✓✓✓✓✓

justify procedures to prove the generalisation ✓✓✓✓✓✓

Question 13 (6 marks)

Representing the examples using ${}^nC_r$ notation:

Row 1: 1 + 2 + 3 = 6 $\Rightarrow {}^1C_0 + {}^2C_1 + {}^3C_2 = {}^4C_2$

Row 4: 1 + 5 + 15 = 21 $\Rightarrow {}^4C_0 + {}^5C_1 + {}^6C_2 = {}^7C_2$

Row 8: 1 + 9 + 45 = 55 $\Rightarrow {}^8C_0 + {}^9C_1 + {}^{10}C_2 = {}^{11}C_2$ ✓✓

Generalising the pattern:

Row n: $\Rightarrow {}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 = {}^{n+3}C_2$ ✓✓✓✓

Proving the generalisation:

LHS $= {}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 = {}^{n+3}C_2$

$= 1+(n+1)+\dfrac{(n+2)!}{(n+2-2)!\,2!}$ ✓

$= n+2+\dfrac{(n+2)!}{n!\,2}$

$= n+2+\dfrac{(n+2)(n+1)n!}{n!\,2}$ ✓

$= \dfrac{2n+4}{2}+\dfrac{(n+2)(n+1)}{2}$

$= \dfrac{2n+4+n^2+3n+2}{2}$

$= \dfrac{n^2+5n+6}{2}$ ✓

RHS $= {}^{n+3}C_2$

$= \dfrac{(n+3)!}{(n+3-2)!\,2!}$ ✓

$= \dfrac{(n+3)!}{(n+1)!\,2}$

$= \dfrac{(n+3)(n+2)(n+1)!}{(n+1)!\,2}$ ✓

$= \dfrac{(n+3)(n+2)}{2}$

$= \dfrac{n^2+5n+6}{2}$ ✓

= LHS

The pattern is proven to be true starting at any row.

14.

analyse the problem for the requirement to construct QS ✓✓

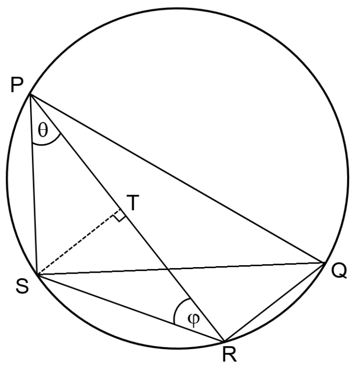
recognise the relationship between PQ and the radius ✓

justify procedures and decisions using mathematical reasoning by:

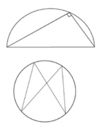
- recognising ∠PSQ is a right angle ✓ and explaining the reason ✓
- expressing ∠PQS in terms of φ ✓✓ and explaining the reason ✓

Question 14 (7 marks)

Constructing chord QS ✓✓



PQ = $2r$ ✓

∠PSQ = 90° ✓ Reason:  ✓

solve the problem by synthesising the information ✓✓✓✓✓✓	∠PQS = φ ✓✓ Reason: ✓ As Δ PQS is right-angled, $\sin\varphi = \frac{\text{PS}}{2r}$ ∴ PS = $2r\sin\varphi$ ………(1) ✓✓ As ΔPTS is right-angled, $\cos\theta = \frac{\text{PT}}{\text{PS}}$ ∴ PT = PS $\cos\theta$ ✓✓ Using (1), PT = $2r\ \cos\theta\ \sin\varphi$ as required. ✓✓	