Mathematical Methods marking guide and response

External assessment 2023

Paper 1: Technology-free (55 marks)

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from Units 3 and 4
2. comprehend mathematical concepts and techniques drawn from Units 3 and 4
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from Units 3 and 4.

Purpose

This document consists of a marking guide and a sample response.

The marking guide:

- provides a tool for calibrating external assessment markers to ensure reliability of results
- indicates the correlation, for each question, between mark allocation and qualities at each level of the mark range
- informs schools and students about how marks are matched to qualities in student responses.

The sample response:

- demonstrates the qualities of a high-level response
- has been annotated using the marking guide.

Mark allocation

Where a response does not meet any of the descriptors for a question or a criterion, a mark of ‘0’ will be recorded.

Where no response to a question has been made, a mark of ‘N’ will be recorded.

Allow FT mark/s — refers to ‘follow through’, where an error in the prior section of working is used later in the response, a mark (or marks) for the rest of the response can still be awarded so long as it still demonstrates the correct conceptual understanding or skill in the rest of the response.

This mark may be implied by subsequent working — the full mathematical reasoning and/or working, as outlined in the sample response and associated mark, is not explicitly stated in the student response, but by virtue of subsequent working there is sufficient evidence to award the mark/s.

Marking guide

Multiple choice

Question	Response
1	C
2	D
3	A
4	C
5	D
6	A
7	D
8	B
9	B
10	C

Short response

Q	Sample response	The response:
11a)	Sample B Mean $= p = \frac{5}{10} = 0.5$ Variance $= p(1-p) = 0.5 \times 0.5$ $= 0.25$	• correctly determines the mean for sample B **[1 mark]** • determines the variance for sample B **[1 mark]**
11b)	Sample B has a larger variance than sample A. Because sample B has the larger variance, it has more variability about the mean compared to sample A.	• identifies that sample B has the larger variance **[1 mark]** • provides reasoning that explains that larger variance indicates larger variability **[1 mark]**

Q	Sample response	The response:
12a)	Let $f(x)=x(4-x)$ $f(1)=1(4-1)=3$ $f(2)=2(4-2)=4$ $f(3)=3(4-3)=3$ $f(4)=4(4-4)=0$ $\sum f(x_i)\delta x_i = (3\times 1)+(4\times 1)+(3\times 1)+(0\times 1)$ $= 10 \text{ metres}^2$	• correctly calculates the described estimation **[1 mark]**
12b)	$\text{Area} = \int_0^4 x(4-x)dx$ $= \int_0^4 (-x^2+4x)\,dx$ $= \left[-\frac{1}{3}x^3+2x^2\right]_0^4$ $= -\frac{1}{3}(4)^3+2(4)^2$ $= -\frac{64}{3}+32$ $= \frac{32}{3}$ $= 10\frac{2}{3}\text{m}^2$	• correctly states the definite integral required **[1 mark]** • determines the integration of the function **[1 mark]** • calculates the area under the curve by integration **[1 mark]**

Q	Sample response	The response:
13a)	One condition is that **there are only two outcomes for each selection**: 'delayed' (success) or 'not delayed' (failure), i.e. Bernoulli trials. Another condition that makes this context suitable is that the **probabilities of each outcome do not change in each trial**, i.e. $p = \frac{1}{5}$ and $q = \frac{4}{5}$	• correctly states one condition for binomial probability **[1 mark]** • correctly states a second condition for binomial probability **[1 mark]**
13b)	$n = 3,\ p = \frac{1}{5},\ 1 - p = \frac{4}{5}$ $P(\text{at least } 2)$ $= P(X = 2 \text{ or } X = 3))$ $= C_2^3 p^2(1-p)^1 + C_3^3 p^3(1-p)^0$ $= 3\left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)^3$ $= 3 \times \frac{1}{25} \times \frac{4}{5} + \frac{1}{5^3}$ $= \frac{12}{125} + \frac{1}{125}$ $= \frac{13}{125}$	• correctly determines the number of trials and the probability of a flight being delayed **[1 mark]** • determines a suitable method **[1 mark]** • calculates the probability **[1 mark]**

Q	Sample response	The response:
14a)	$P(t) = \int e^{2t}dt$ $= \frac{1}{2}e^{2t} + c$ $P(0) = 60$ $60 = \frac{1}{2}e^{2(0)} + c$ $c = 59.5$ $P(t) = \frac{1}{2}e^{2t} + 59.5$	• correctly determines the integral from the rate **[1 mark]** • determines the value of *c* **[1 mark]**
14b)	The total change in the number of bacteria during the third hour ($t = 2$ to $t = 3$) is: $P(3) - P(2)$ $= \frac{1}{2}e^{2\times 3} + 59.5 - \left(\frac{1}{2}e^{2\times 2} + 59.5\right)$ $= \frac{1}{2}e^{6} - \frac{1}{2}e^{4}$ bacteria	• identifies the method required for the third hour calculation **[1 mark]** • determines an expression for the total change **[1 mark]**
14c)	$60 \times 2 = 120$ $120 = \frac{1}{2}e^{2t} + 59.5$ $60.5 = \frac{1}{2}e^{2t}$ $121 = e^{2t}$ $\ln 121 = 2t$ $t = \frac{1}{2}\ln 121$	• correctly determines the doubled population **[1 mark]** • determines expression for the time required **[1 mark]**

Q	Sample response	The response:
15a)	There are two outcomes: hit or miss. Hit $P(\text{hit}) = \frac{1}{4}$ Miss $P(\text{miss}) = \frac{3}{4}$	• correctly states the probability of a 'hit' **[1 mark]** • correctly states the probability of a 'miss' **[1 mark]**
15b)	Using *H* to represent the number of 'hits'. The mean of a binomial distribution is np. $np = 20 \times \frac{1}{4} = 5$ The variance of a binomial distribution is $np(1-p)$. $np(1-p) = 5 \times \frac{3}{4}$ $= \frac{15}{4} = 3\frac{3}{4}$	• determines the mean **[1 mark]** • determines the variance **[1 mark]**

Q	Sample response	The response:
16	$4+7e^{-2x}=3e^{2x}$ $4e^{2x}+7=3e^{4x}$ $3e^{4x}-4e^{2x}-7=0$ $(3e^{2x}-7)(e^{2x}+1)=0$ $3e^{2x}-7=0$ $\quad$ $e^{2x}+1=0$ $3e^{2x}=7$ $\quad$ $e^{2x}=-1$ $e^{2x}=\frac{7}{3}$ $\quad$ not possible $\ln\left(\frac{7}{3}\right)=2x$ $x=\frac{1}{2}\ln\left(\frac{7}{3}\right)$	• correctly removes the negative index in the equation **[1 mark]** • rearranges equation to equate to zero **[1 mark]** • factorises the equation **[1 mark]** • rejects the non-feasible solution **[1 mark]** • determines a feasible solution for x **[1 mark]**

Q	Sample response	The response:
17	The rate of absorption is given by: $\frac{dA}{dt} = 20t - 12t^2$	• correctly determines the first derivative **[1 mark]**
	$\frac{d^2A}{dt^2} = 20 - 24t$	• determines the second derivative **[1 mark]**
	$\therefore 20 - 24t = 0$ $t = \frac{20}{24} = \frac{5}{6}$ hours	• equates the second derivative to zero **[1 mark]** • determines time when second derivative is zero **[1 mark]**
	Verify this corresponds to a maximum rate. Using the second derivative test, we investigate the sign of the derivative of $\frac{d^2A}{dt^2}$, i.e. $\frac{d^3A}{dt^3}$ $\frac{d^3A}{dt^3} = -24$ This is negative, therefore the rate of absorption is a maximum.	• performs a calculus test to confirm the time corresponds to a maximum for $\frac{dA}{dt}$ **[1 mark]**
	The time the chemical is increasing most rapidly since delivery is $\frac{5}{6}$ hours. $= \frac{5}{6} \times 60$ $= 50$ minutes The required time is 10:50 am.	• determines the time for maximum rate of absorption in minutes **[1 mark]**

Q	Sample response	The response:
18	Because of the repetitive nature of the motion, use a cosine graph (cosine since when t = 0, the graph is a minimum) to model the motion. $h(t)=a\cos(b(t+c))+d$ Amplitude = 3 $\therefore a=3$ Period = 6 seconds $\therefore T=\frac{2\pi}{b}$ $b=\frac{2\pi}{6}=\frac{\pi}{3}$ No phase shift $\therefore c=0$ The axis of the Ferris wheel is 4 m above ground $\therefore d=4$	• correctly recognises a periodic model is to be used for the vertical position of the carriage **[1 mark]**
	$h(t)=-3\cos\left(\frac{\pi}{3}t\right)+4$ $v(t)=h'(t)=3\frac{\pi}{3}\sin\left(\frac{\pi}{3}t\right)=\pi\sin\left(\frac{\pi}{3}t\right)$ $a(t)=h''(t)=\pi\frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right)=\frac{\pi^2}{3}\cos\left(\frac{\pi}{3}t\right)$ The amplitude of the acceleration model is $\frac{\pi^2}{3}$. This also corresponds to the maximum value of acceleration.	• correctly determines the model for the vertical position of a carriage on the Ferris wheel **[1 mark]**
	$\frac{\pi^2}{3}\approx\frac{3^2}{3}=3$	• determines an approximation of the maximum acceleration produced based on the second derivative equation obtained **[1 mark]**
	Therefore, it is not reasonable to expect the acceleration of the Ferris wheel alone to be more than half of 9.8 (4.9).	• provides appropriate statement of reasonableness **[1 mark]**

Q	Sample response	The response:
19	For each shop use $y = A\ln(Bx)$ as the model. Find the constants A and B for each shop. Shari's shop $10 = A\ln B$ .... (1) $20 = A\ln(2B)$ .... (2) $20 = A\ln 2 + A\ln B$ $20 = A\ln 2 + 10$ from (1) $A\ln 2 = 10$ $A = \frac{10}{\ln 2}$ Substitute into (1) $10 = \frac{10}{\ln 2}\ln B$ $\ln B = \frac{10\ln 2}{10}$ $\ln B = \ln 2$ $B = 2$ $y = A\ln Bx$ $y = \left(\frac{10}{\ln 2}\right)\ln(2x)$ ... Shari's shop model Jaxon's shop $40 = A\ln B$ .... (1) $30 = A\ln(2B)$ .... (2) $30 = A\ln 2 + A\ln B$ $30 = A\ln 2 + 40$ from (1) $A\ln 2 = -10$ $A = \frac{-10}{\ln 2}$ Substitute into (1) $40 = \frac{-10}{\ln 2}\ln B$ $\ln B = \frac{40\ln 2}{-10}$	• correctly determines A (or B) for Shari's shop **[1 mark]** • determines the model for Shari's shop **[1 mark]** • correctly determines A (or B) for Jaxon's shop **[1 mark]**

Q	Sample response	The response:
	$\ln B = \left(\frac{40}{-10}\right)\ln 2$ $\ln B = -4\ln 2$ $\ln B = \ln 2^{(-4)}$ $\therefore B = 2^{-4} = \frac{1}{16}$ $y = A\ln(Bx)$ $y = \left(\frac{-10}{\ln 2}\right)\ln(\frac{1}{16}x)$ … Jaxon's shop model	• determines the model for Jaxon's shop **[1 mark]**
	On any day, x, the sum of the two models is: $= \left(\frac{10}{\ln 2}\right)\ln(2x) + \left(\frac{-10}{\ln 2}\right)\ln(\frac{1}{16}x)$	• determines an expression for the total number of daily customers in both shops **[1 mark]**
	$= \left(\frac{10}{\ln 2}\right)\ln(2x) - \left(\frac{10}{\ln 2}\right)\ln(\frac{1}{16}x)$ $= \left(\frac{10}{\ln 2}\right)[\ln(2x) - \ln(\frac{1}{16}x)]$ $= \left(\frac{10}{\ln 2}\right)\ln(\frac{2x}{\frac{1}{16}x})$ $= \left(\frac{10}{\ln 2}\right)\ln(\frac{2}{\frac{1}{16}})$ $= \left(\frac{10}{\ln 2}\right)\ln(32)$ $= \frac{10\ln 32}{\ln 2}$ $= \frac{10\ln(2^5)}{\ln 2}$ $= \frac{10\times 5\ln(2)}{\ln 2}$ = 50	• justifies the sum obtained by explaining mathematical reasoning **[1 mark]**
	The sum of the models does result in the same number of customers, i.e. 50 on any day.	• provides an appraisal by interpreting the result of the analysis of the sum of the two models performed **[1 mark]**