Mathematical Methods marking guide

External assessment 2021

Paper 1: Technology-free (55 marks)

Paper 2: Technology-active (55 marks)

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from Units 3 and 4
2. comprehend mathematical concepts and techniques drawn from Units 3 and 4
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from Units 3 and 4.

Queensland Government
Queensland Curriculum & Assessment Authority
For all Queensland schools

Purpose

This marking guide:

- provides a tool for calibrating external assessment markers to ensure reliability of results
- indicates the correlation, for each question, between mark allocation and qualities at each level of the mark range
- informs schools and students about how marks are matched to qualities in student responses.

Mark allocation

Where a response does not meet any of the descriptors for a question or a criterion, a mark of ‘0’ will be recorded.

Where no response to a question has been made, a mark of ‘N’ will be recorded.

Allowing for FT mark/s — refers to ‘follow through’, where an error in the prior section of working is used later in the response, a mark (or marks) for the rest of the response can be awarded so long as it still demonstrates the correct conceptual understanding or skill in the rest of the response.

This mark may be implied by subsequent working — the full mathematical reasoning and/or working, as outlined in the sample response and associated mark, is not explicitly stated in the student response, but by virtue of subsequent working there is sufficient evidence to award the mark/s.

Marking guide

Paper 1

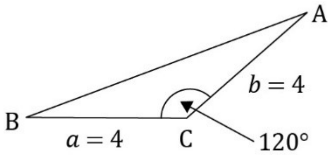
Multiple choice

Question	Response
1	A
2	B
3	C
4	C
5	D
6	A
7	C
8	B
9	B
10	C

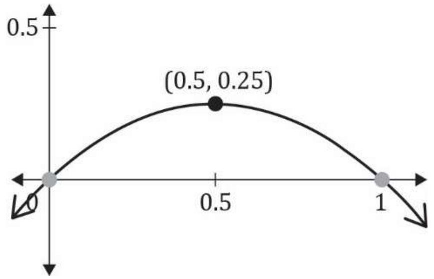
Short response

Q	Sample response	The response:
11a)	Let $u = e^x + 1$ Using the chain rule $\frac{dy}{dx} = 3e^x(e^x + 1)^2$	• correctly identifies the use of the chain rule **[1 mark]** • correctly determines the derivative **[1 mark]**
11b)	Using the quotient rule $\frac{dy}{dx} = \frac{x^2\cos(x) - \sin(x)2x}{(x^2)^2}$ $\frac{dy}{dx} = \frac{x\cos(x) - 2\sin(x)}{x^3}$	• correctly identifies the use of the quotient rule **[1 mark]** • correctly determines the derivative **[1 mark]** • provides derivative in simplest form **[1 mark]**
12a)	Changing from log to index form $5x + 7 = 32$ $5x = 25$ $x = 5$	• correctly establishes the linear equation **[1 mark]** • determines x **[1 mark]**
12b)	Using addition log law $\log_{10}(x^2 - 9) = \log_{10}(9x - 29)$ Equating and rearranging $x^2 - 9x + 20 = 0$ Factorising $(x - 4)(x - 5) = 0$ $\rightarrow x = 4, 5$	• correctly applies the log law **[1 mark]** • establishes quadratic equation **[1 mark]** • determines 2 solutions **[1 mark]**

Q	Sample response	The response:
13a)	Solving simultaneously $x^2 = 4x$ Rearranging and factorising $x(x-4) = 0$ $\therefore x = 0$ and $x = 4$	• correctly uses the simultaneous procedure **[1 mark]** • correctly determines both the x-intercept ordinates **[1 mark]**
13b)	Area = $\int_0^4 (4x - x^2)dx$ $= 2x^2 - \frac{x^3}{3}\Big\vert_0^4$ $= \left(2 \times 4^2 - \frac{4^3}{3}\right) - 0$ $= \frac{32}{3}$ square units	• correctly determines the integral **[1 mark]** • substitutes limits into integral **[1 mark]** • determines area **[1 mark]**
14a)	$f'(x) = \frac{3}{(3x+4)}$	• correctly determines the derivative **[1 mark]**
14b)	x-intercept $(y = 0)$ $0 = \ln(3x+4)$ $3x + 4 = 1$ $x = -1$	• correctly determines the linear equation **[1 mark]** • correctly determines the x-intercept **[1 mark]**
14c)	$f'(-1)$ $= \frac{3}{-3+4}$ $= 3$	• determines the gradient at the x-intercept **[1 mark]**

Q	Sample response	The response:
15a)		• correctly labels the given angle and sides in the isosceles triangle **[1 mark]**
15b)	Area $= \frac{1}{2} \times 4 \times 4 \times \sin 120°$ $= \dfrac{1}{2} \times 4 \times 4 \times \dfrac{\sqrt{3}}{2}$ Area $= 4\sqrt{3}$ cm^2	• establishes expression for the area **[1 mark]** • correctly determines the exact value of sine **[1 mark]** • determines area in simplest form **[1 mark]**
16	$y' = -e^{2-x}$ Gradient of tangent at $(1, e)$ $y'(1) = -e$ Equation of the tangent $(y - e) = -e(x - 1)$ x-intercept $(2,0)$ y-intercept $(0,2e)$ Area of triangle = $\frac{1}{2} \times 2 \times 2e$ Area of triangle = $2e$ units^2	• correctly determines the derivative **[1 mark]** • determines equation of tangent **[1 mark]** • determines x- and y-intercepts **[1 mark]** • determines area of triangle **[1 mark]**

Q	Sample response	The response:
17	Using binomial distribution $n = 5$ $p = \frac{3}{5}$ and $q = \frac{2}{5}$ $P(\text{catches the bus on 1 day})$ $= \binom{5}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^4$ $= 3 \times \frac{16}{5^4} = \frac{48}{625}$	• correctly determines the values for p and n **[1 mark]** • establishes expression for the required probability **[1 mark]** • determines probability **[1 mark]**
18	$\int_{-1}^{0} f(x)dx = \frac{13}{6}$ $\int_{0}^{1} f(x)dx = \frac{43}{6}$ $\frac{ax^3}{3} + \frac{bx^2}{2} + 4x + c \Big\vert_{-1}^{0} = \frac{13}{6}$ (i) $\frac{ax^3}{3} + \frac{bx^2}{2} + 4x + c \Big\vert_{0}^{1} = \frac{43}{6}$ (ii) From (i) $0 - \left(\frac{-a}{3} + \frac{b}{2} - 4\right) = \frac{13}{6}$ $2a - 3b = -11$ (A) From (ii) $2a + 3b = 19$ (B) (A)–(B) $-6b = -30$ $b = 5$ Sub into (A) $2a = 4$ $a = 2$ $\therefore f(x) = 2x^2 + 5x + 4$	• correctly identifies the use of integrals **[1 mark]** • correctly determines the two equations in the two unknowns a and b **[1 mark]** • determines b **[1 mark]** • determines a **[1 mark]**

Q	Sample response	The response:
19	$interval\ margin = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $0.05 = 2\sqrt{\frac{0.5(1-0.5)}{n}}$ rearranging $n = \frac{2^2 \times 0.5(0.5)}{(0.05)^2}$ $n = 400$ The largest sample size will result when $\hat{p}(1-\hat{p})$ is maximised in the numerator, therefore generating largest n value.  Maximum occurs at $\hat{p} = 0.5$	• correctly selects the interval margin formula **[1 mark]** • substitutes values into the formula **[1 mark]** • determines sample size n **[1 mark]** • verifies the firm's decision to use $\hat{p} = 0.5$ using mathematical reasoning **[1 mark]**

Q	Sample response	The response:
20	**Method 1** To determine intervals $P'(t) = t + 2t \times \ln(3t)$	• correctly determines $P'(t)$ **[1 mark]**
	Critical points $P'(t) = 0$ $0 = t(1 + 2\ln(3t))$ $t = 0$ (reject)	• correctly determines the rejected solution for t **[1 mark]**
	$1 + 2\ln(3t) = 0 \rightarrow t = \dfrac{1}{3\sqrt{e}}$	• determines t-ordinate of critical point **[1 mark]**
	To determine the nature of $t = \frac{1}{3\sqrt{e}}$ $P''(t) = 3 + 2\ln(3t)$ $P''\left(\dfrac{1}{3\sqrt{e}}\right) = 3 + 2\ln\left(\dfrac{1}{\sqrt{e}}\right)$ $P''\left(\dfrac{1}{3\sqrt{e}}\right) = 2$	• determines value of $P''(t)$ at critical point **[1 mark]**
	∴ the point is a minimum.	• determines nature of critical point **[1 mark]**
	Therefore the population decreases for $0 < t < \frac{1}{3\sqrt{e}}$ but increases for $t > \frac{1}{3\sqrt{e}}$	• communicates when the population is increasing and when it is decreasing **[1 mark]**
		• shows logical organisation communicating key steps **[1 mark]**

Q	Sample response	The response:
20	**Method 2** To determine intervals $P'(t) = t + 2t \times \ln(3t)$	
	$P'(t) = t(1 + 2\ln(3t))$	• correctly determines $P'(t)$ **[1 mark]**
	Population is increasing when $P'(t) > 0$ $t(1 + 2\ln(3t)) > 0$	• correctly identifies the time interval required to determine increasing population **[1 mark]**
	Given $t > 0$	• identifies relevance of domain **[1 mark]**
	$\therefore (1 + 2\ln(3t)) > 0$	• establishes inequality in t **[1 mark]**
	$\ln(3t) > \frac{-1}{2}$ $t > \frac{1}{3\sqrt{e}}$	• determines interval when population is increasing **[1 mark]**
	$\therefore$ the population decreases for $0 < t < \frac{1}{3\sqrt{e}}$	• determines interval when population is decreasing **[1 mark]**
		• shows logical organisation communicating key steps **[1 mark]**

Marking guide

Paper 2

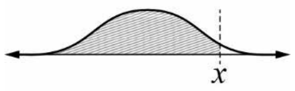
Multiple choice

Question	Response
1	A
2	A
3	D
4	C
5	D
6	A
7	B
8	D
9	D
10	A

Short response

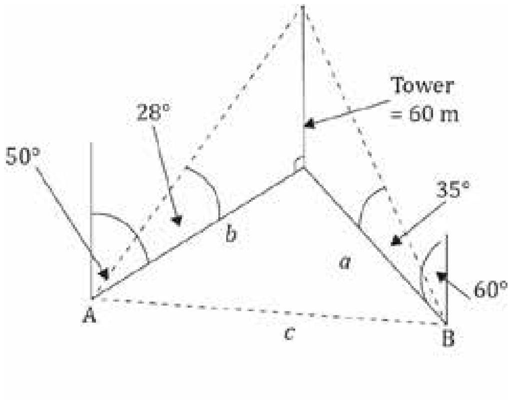
Q	Sample response	The response:
11a)	x-intercepts $(y = 0)$ $0 = e^x \sin(x)$ Using null factor rule $e^x \neq 0$ $\sin(x) = 0$ $\therefore x = 0,\ \pi,\ 2\pi$	· correctly generates the equation required **[1 mark]** · correctly determines the three x-intercepts **[1 mark]**
11b)	$\int_0^{\pi} e^x \sin(x)dx + \left\lvert \int_{\pi}^{2\pi} e^x \sin(x)dx \right\rvert$	· determines expression for integral above the x-axis **[1 mark]** · determines expression for integral below the x-axis (including absolute value brackets) **[1 mark]**
11c)	Area enclosed (using GDC) 291 square units	· determines area to the nearest unit **[1 mark]**
12a)	$s(t) = \frac{1}{6}\sin\left(6t + \frac{\pi}{2}\right) + 2t + c$ Substituting $(0,\ 0)$ $0 = \frac{1}{6}\sin\left(\frac{\pi}{2}\right) + c$ $c = \frac{-1}{6}$ $s(t) = \frac{1}{6}\sin\left(6t + \frac{\pi}{2}\right) + 2t + \frac{-1}{6}$	· correctly determines the indefinite integral $s(t)$ **[1 mark]** · correctly determines the displacement function **[1 mark]**
12b)	distance = $s(3) - s(0)$ $= \frac{1}{6}\sin\left(18 + \frac{\pi}{2}\right) + 6 - \frac{1}{6} - \left(\frac{1}{6}\sin\left(\frac{\pi}{2}\right) - \frac{1}{6}\right)$ = 5.943 m	· establishes an expression for the distance travelled **[1 mark]** · determines distance travelled **[1 mark]**

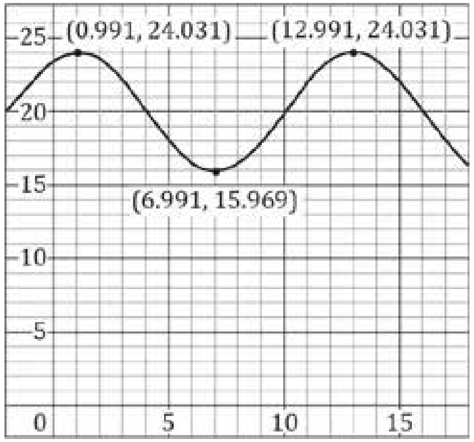
Q	Sample response	The response:
13a)	Using GDC $\int_0^1 \frac{3}{2}(1-x^2)dx = 1$	· correctly establishes the definite integral equated to 1 **[1 mark]**
13b)	$P(X < 0.25)$ $= \int_0^{0.25} \frac{3}{2}(1-x^2)dx$ $= 0.367$	· correctly establishes the definite integral **[1 mark]** · correctly determines the probability **[1 mark]**
13c)	Mean $= \int_0^1 \frac{3}{2}x(1-x^2)\,dx$ $= 0.375$ Variance $= \int_0^1 \frac{3}{2}(x-0.375)^2(1-x^2)\,dx$ $= 0.059$ tonnes^2	· correctly establishes the definite integral **[1 mark]** · correctly establishes the mean **[1 mark]** · establishes definite integral **[1 mark]** · determines variance **[1 mark]**

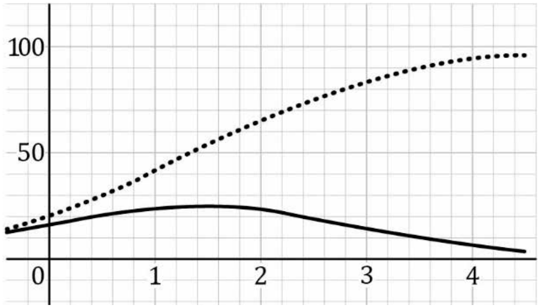
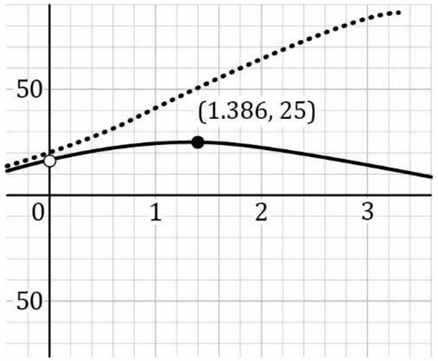
Q	Sample response	The response:
14a)	Using GDC $P(\text{student height under } 180\text{ cm})$ $= 0.841$	· correctly determines the probability **[1 mark]**
14b)	 	· correctly uses an appropriate mathematical representation **[1 mark]**
	$x = 195.806$	· correctly determines the lowest decimal height **[1 mark]**
	∴ minimum height is 196 cm	· determines lowest whole height **[1 mark]**
14c)	$z_{\text{School A}} = \frac{196 - 165}{15}$ $z_{\text{School A}} = \frac{31}{15}$	· determines z-score for student in school A **[1 mark]**
	$\frac{31}{15} < 3$	· provides statement to justify decision **[1 mark]**
	∴ student in School B ranked higher	· determines higher ranked student **[1 mark]**

Q	Sample response	The response:
15	**Method 1** Current ranking $= \log_{10}(50 \times 100^2)$ $= 5.69897$ ∴ increased ranking is 6.69897 $6.69897 = \log_{10}(50h^2)$ $h = 316.228$ ∴ the website requires an additional 217 hits to increase their ranking by 1.	· correctly determines the website's increased ranking **[1 mark]** · determines number of hits for increased ranking **[1 mark]** · provides reasonable solution for number of additional hits **[1 mark]** · shows logical organisation communicating key steps **[1 mark]**

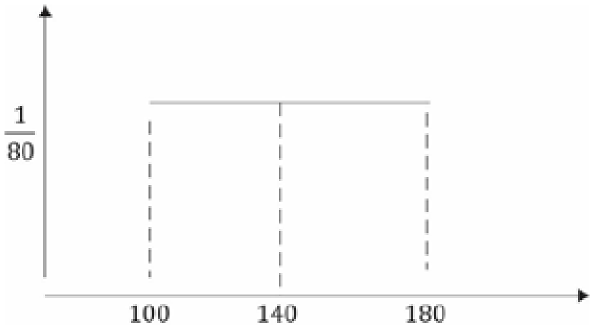
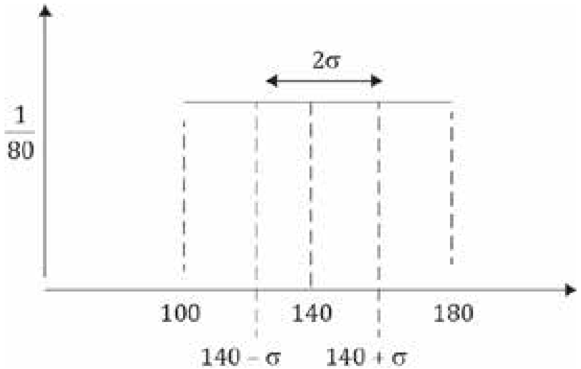
Q	Sample response	The response:
15	**Method 2** $R_{old} = \log_{10}(50h^2)$ $R_{old} = \log_{10}(50) + 2\log_{10}(h)$ Let k = number of additional hits $R_{new} = \log_{10}(50) + 2\log_{10}(h+k)$ $R_{new} = R_{old} + 1$ $\log_{10}(50) + 2\log_{10}(h+k) = \log_{10}(50) + 2\log_{10}(h) + 1$ $2\log_{10}(h+k) = 2\log_{10}(h) + 1$ $2\log_{10}(h+k) - 2\log_{10}(h) = 1$ $\log_{10}\frac{(h+k)}{h} = \frac{1}{2}$ $\frac{(h+k)}{h} = \sqrt{10}$ Currently $h = 100$ $k = 216.228$ ∴ the website requires an additional 217 hits to increase their ranking by 1.	· correctly determines the equation using increased ranking of 1 **[1 mark]** · determines number of additional hits for increased ranking **[1 mark]** · provides reasonable solution for number of additional hits **[1 mark]** · shows logical organisation communicating key steps **[1 mark]**

Q	Sample response	The response:
16	 Distance from A to the tower (b): $\tan 28° = \frac{60}{b}$ Distance from B to the tower (a): $\tan 35° = \frac{60}{a}$ $b = 112.84$ m and $a = 85.69$ m Distance between A and B (c) $c^2 = 85.69^2 + 112.84^2 - 2 \times 85.69 \times 112.84 \times \cos 110°$ $c = 163.37$ m The distance between A and B is 163 m.	· correctly establishes the angles using given bearings **[1 mark]** · correctly determines the distance a and b **[1 mark]** · determines distance between A and B **[1 mark]** · shows logical organisation communicating key steps **[1 mark]**

Q	Sample response	The response:
17	Models are periodic. Rabbits: $R(t) = 11 + 3.5\cos\left(\frac{\pi}{6}t\right)$ Foxes: $F(t) = 9 + 2\sin\left(\frac{\pi}{6}t\right)$	· correctly determines the models for the two populations **[1 mark]**
	Total population of foxes and rabbits $T(t) = 20 + 3.5\cos\left(\frac{\pi}{6}t\right) + 2\sin\left(\frac{\pi}{6}t\right)$	· determines model for total population of foxes and rabbits **[1 mark]**
	Graphing $T(t)$ 	· uses an appropriate mathematical representation **[1 mark]**
	Greatest total population of foxes and rabbits is 24 031. Jane's claim is not correct. The maximum total population occurs on one occasion in the year, but does not exceed 25 000 (maximum is 24 031).	· evaluates reasonableness of the claim **[1 mark]**

Q	Sample response	The response:
18	Graphing $P(t)$ (dotted line) and $P'(t)$ (solid line)  The population is increasing most rapidly at the maximum value of $P'(t)$  à $t = 1.386$ $\therefore P(1.386) \sim 49.993$ There are approximately 50 000.	· correctly identifies the conditions for the most rapid increase **[1 mark]** · determines when population growing the fastest **[1 mark]** · determines population at this time **[1 mark]**

Q	Sample response	The response:
19	**Method 1** If X denotes the number of minutes that elapse between the placement and delivery of the order, then X can take any value in the interval $100 \leq X \leq 180$. Since the width is 80, the height of the density function must be $\frac{1}{80}$ for the total area under the density function to equal 1. The probability density function is: $f(x) = \frac{1}{80}, 100 \leq x \leq 180$ Using the given rules $E(X) = \frac{180 + 100}{2}$ $\text{Var}(X) = \frac{80^2}{12} = \frac{1600}{3}$ $\therefore$ Mean $= 140$ and standard deviation $= \frac{40}{\sqrt{3}}$ Required probability $P(140 - \frac{40}{\sqrt{3}} < X < 140 + \frac{40}{\sqrt{3}})$ $= \int_{140 - \frac{40}{\sqrt{3}}}^{140 + \frac{40}{\sqrt{3}}} \frac{1}{80}\, dx$ $= 0.57735$	· correctly determines the probability density function **[1 mark]** · correctly determines the mean and the standard deviation **[1 mark]** · establishes definite integral to represent probability **[1 mark]** · determines probability **[1 mark]**

Q	Sample response	The response:
19	**Method 2** Graphing the probability density function:  Using the given rules $E(X) = \frac{180 + 100}{2}$ $\text{Var}(X) = \frac{80^2}{12} = \frac{1600}{3}$ $\therefore \text{Mean } = 140$ and standard deviation $= \frac{40}{\sqrt{3}}$ Identifying required probability as an area  Required probability $= \frac{1}{80} \times 2 \times \frac{40}{\sqrt{3}}$ Required probability $= \frac{1}{\sqrt{3}}$	· correctly identifies the probability density function graphically **[1 mark]** · correctly determines the mean and the standard deviation **[1 mark]** · uses appropriate graphical representation **[1 mark]** · determines probability **[1 mark]**

Q	Sample response	The response
20	The quadratic has real roots when $b^2 - 4ac \geq 0$ $\therefore 9 - 8B \geq 0$ $\therefore 9 \geq 8B$ $\therefore \frac{9}{8} \geq B$ Using the standard normal distribution (the given distribution) $P(B \leq \frac{9}{8})$ $= 0.8697$	· correctly identifies the need to use the discriminant **[1 mark]** · correctly determines the range of values for B **[1 mark]** · determines probability **[1 mark]**