Mathematical Methods 2019 v1.2

Unit 1 Topic 2 high-level annotated response

September 2018

Problem-solving and modelling task

This sample has been compiled by the QCAA to assist and support teachers to match evidence in student responses to the characteristics described in the assessment objectives.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from Unit 1 Topic 2
2. comprehend mathematical concepts and techniques drawn from Unit 1 Topic 2
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from Unit 1 Topic 2.

Task

Context
Computer-generated images in video games and film and TV special effects are created using a number of mathematical concepts and techniques, including basic arithmetic, geometry, trigonometry, linear algebra and calculus. A computer animation studio is interested in developing its own ‘physics engine’. They have asked you to work on one aspect of the engine — how free-hanging objects act under the influence of gravity. It has been suggested that as a proof of concept, a function can be used to model the shape of a hanging chain.
Task
Consider a flexible chain of length, l, that hangs freely. The ends of the chain are secured at two fixed points that are at the same height and are distance, d, apart. Develop a function that models the shape of the hanging chain, and then produce a report that explains how you developed and refined your model. You must consider: • a polynomial function • the sum of a number of even-degree polynomials described as: $y = a_0 + b_2(x - a_2)^2 + b_4(x - a_4)^4 + \cdots + b_{2n}(x - a_{2n})^{2n} + \cdots$.

Sample response

Criterion	Marks allocated	Result
Formulate Assessment objectives 1, 2, 5		
Solve Assessment objectives 1 and 6		
Evaluate and verify Assessment objectives 4 and 5		
Communicate Assessment objective 3		
Total		

Communicate coherent and concise organisation of the response The introduction describes what the task is about and briefly outlines how the writer intends to complete the task. **Formulate** accurate translation of all aspects of the problem by identifying mathematical concepts and techniques	## Introduction In this report, the model that best describes the shape of a freely hanging chain of length 433 mm, supported at two ends that are 180 mm apart is determined. To carry out the investigation, a suitable frame of reference data is used to determine the primary data needed to generate feasible mathematical models. Both polynomial models and a real-valued polynomial model consisting of the sum of a number of even-degree polynomials of the type $y = a_0 + b_2(x - a_2)^2 + b_4(x - a_4)^4 + \cdots + b_{2n}(x - a_{2n})^{2n} + \cdots$ are considered. Both technological and mathematical procedures are used to find the models, including transformations, simultaneous equation solving and technology, such as Excel and Desmos. The feasibility of the models is tested by considering measures including the correlation coefficient and residual analysis. Recommendations as to the usefulness of models are also discussed.
	## Method A 1 mm × 1 mm sheet of graph paper was attached with adhesive to a large fixed window. It was important that the graph paper's grid lines were aligned vertically and horizontally. The experimental set-up is shown in Figure 1 below.  Figure 1: A photograph showing the experimental set-up with the graph paper stuck to the window and the freely hanging chain.
Formulate accurate translation of all aspects of the problem by identifying mathematical concepts and techniques	The bottom left-hand corner of the graph paper was chosen as the origin of the Cartesian plane. The chain was affixed very precisely to the graph paper at coordinates $(0, 270)$ and $(180, 270)$. The domain for the model will therefore be $0 \leq x \leq 270$. A fine needle was inserted through the hollow centres of the chain links to pierce the graph paper beneath. Care was taken to avoid contact with the chain to ensure there was no effect on its free hanging position, ensuring accurate data was collected.
Formulate accurate translation of all aspects of the problem identifying mathematical concepts and techniques; accurate documentation of relevant observations	Needle pricks were made in the graph paper approximately every half centimetre along the chain length, giving 35 data points. Since the chain hangs freely under its own weight, the curve is symmetrical about the vertical line through its apex or turning point. It was decided, nonetheless, to collect data points along the entire length of the chain. This allowed the data set to be doubled (by reflecting the data points in the vertical line through the turning point). The 35 needle pricks were then converted into Cartesian coordinates relative to the origin.

Communicate
coherent and concise organisation of the response

The data points are tabulated in Figure 2.

Cartesian coordinates of a freely hanging chain					
x (mm)	y (mm)	x (mm)	y (mm)	x (mm)	y (mm)
0	270	63	96	150	151.5
4	244.5	71.5	90.5	154	164.5
7.5	230	78.5	88	159	180.5
10.5	214.5	85.5	86.5	163	195
13.5	200.5	93.5	86.5	167	210.5
17.5	186.5	100.5	89	169	219
22	169.5	113	95	171	230.5
27	155.5	119	100.5	173	237.5
36	132.5	127	109	175	248
43	121.5	134	120	177	260.5
47.5	113.5	139	127.5	180	270
57	102	144.5	136.5		

Figure 2: Tabulated (x, y) position data for a freely hanging chain.

Analysis

Communicate
coherent and concise organisation of the response

Figure 3 below shows a plot of the data using technology.

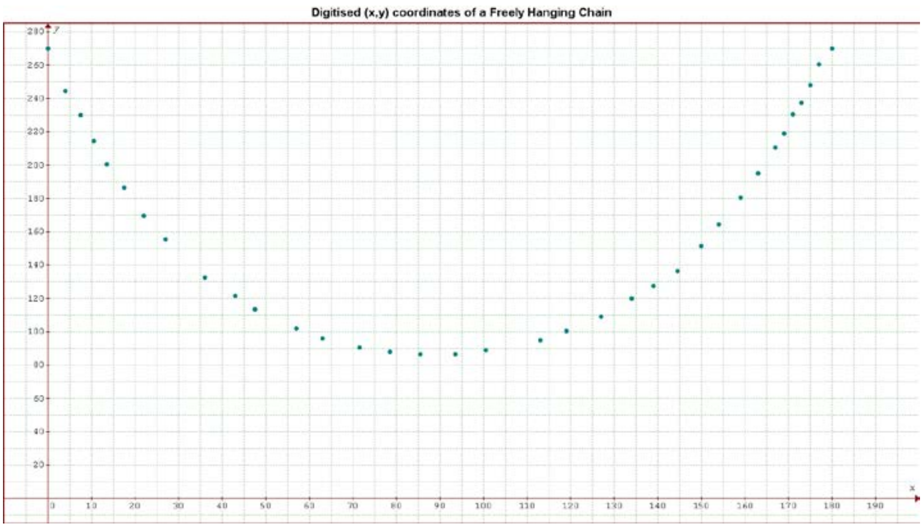
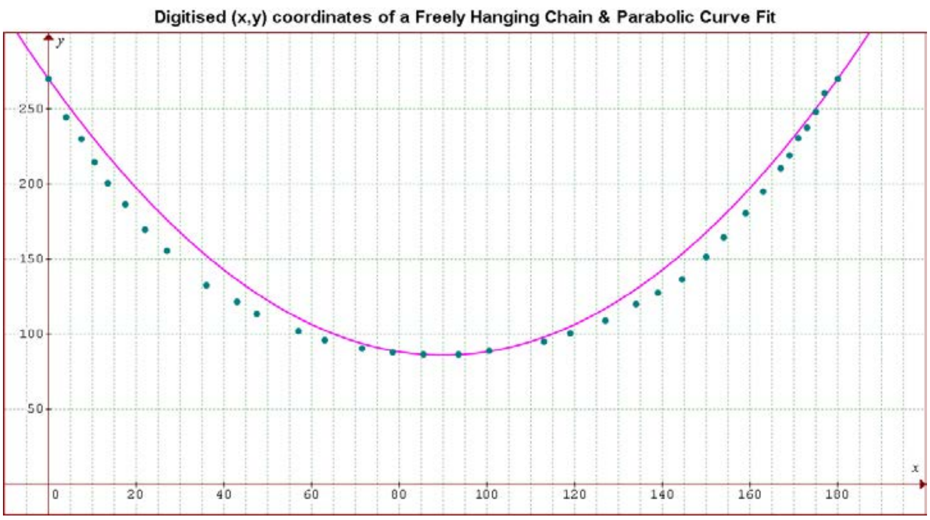


Figure 3: Graph showing the (x, y) position data for a freely hanging chain. All dimensions are in mm.

Formulate
documentation of appropriate assumptions

It was observed the sketch resembled a parabola in shape. It was therefore assumed that a quadratic function would provide a reasonable model. This was investigated.

Using coordinates:

accurate documentation of relevant observations	Left fixation coordinate $(0, 270)$ Right fixation coordinate $(180, 270)$ Turning point $\left(\frac{180}{2}, 86\right) = (90, 86)$ Basic quadratic function: $y = x^2$ Turning point $(0, 0)$ Transformed quadratic function $y = a(x + b)^2 + c$ (1)
Solve discerning application of mathematical concepts and techniques relevant to the task Student determines the function by shifting vertically and horizontally and by stretching or compressing.	The turning point of the basic function is $(0, 0)$, while the turning point of the transformed function is $(90, 86)$. This means that the curve has been translated in the positive y-direction by 86 mm and translated in the positive x-direction by 90 mm; therefore, the parameter values of c and b are 86 and -90 respectively. Substituting these values into equation (1) gives: $y = a(x - 90)^2 + 86$ (2) The point $(0, 270)$ lies on the curve. Substituting $(0, 270)$ into equation (2): $270 = a(0 - 90)^2 + 86$ and re-arranging: $270 - 86 = a \times 90^2$ $184 = a \times 90^2$ $a = \frac{184}{8100}$ $a = \frac{46}{2025}$ The proposed parabolic function to model the position data of the freely hanging chain is: $y = \frac{46}{2025}(x - 90)^2 + 86$ " " model (A) The graph of model (A) and the raw position data is shown in Figure 4 below.  Figure 4: Graph showing the (x, y) position data of a freely hanging chain and model (A) determined analytically. All dimensions are in mm.

Formulate accurate documentation of relevant observations **Evaluate and verify** evaluation of the reasonableness of solutions by considering the results, assumptions and observations	It was observed that the curve made by a freely hanging chain is flatter than the parabolic function. While the parabola fits the data at the two fixation points and the turning point (since that was how the parabolic function was derived), it is too narrow everywhere else. Using Excel, the best fitting parabolic function was also found ($y = 0.0224x^2 - 3.9874x + 255.48$ Model (B)). The graph of Model (B) is shown in Figure 5. Even though the R^2 value is 0.9887 which indicates a very strong positive correlation, it is clear from the graph that the fit is also not appropriate.

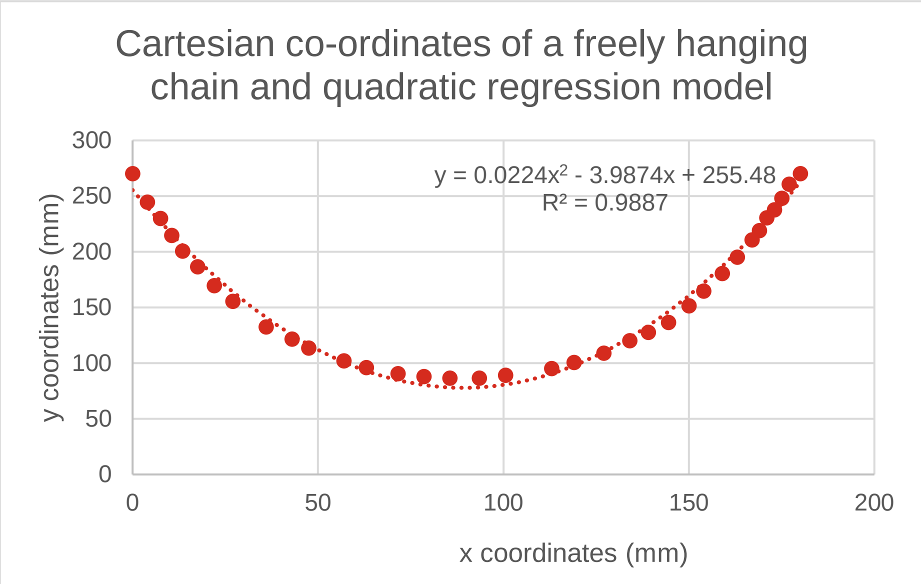


Figure 5: Graph showing the *(x, y)* position data of a freely hanging chain and technology-generated quadratic Model (B).

The analysis was repeated using technology and the quartic regression model produced a higher R^2 value of 0.9995. The quartic model is shown in Figure 6.

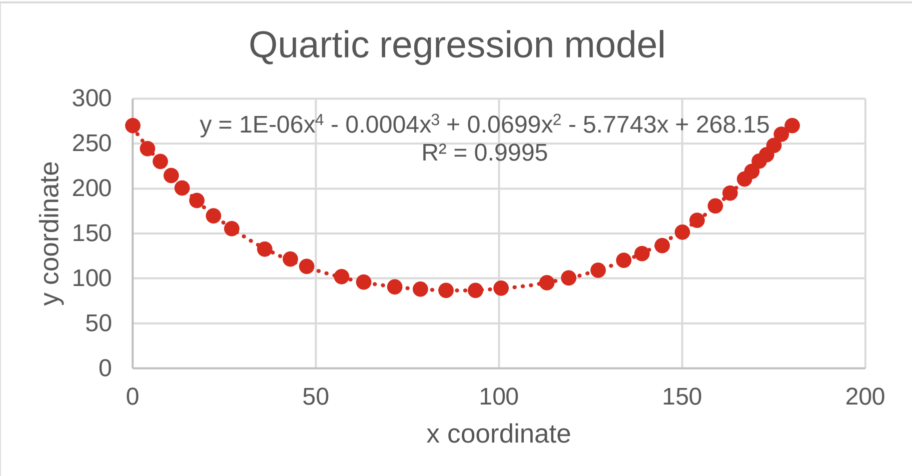


Figure 6: Graph showing the *(x, y)* position data of a freely hanging chain and Excel's best fitting quartic polynomial model.

Formulate accurate translation of all aspects of the problem by identifying mathematical concepts and techniques	The viability of all models was further investigated using a residual analysis, which showed the variation between the observed *y*-value data set and the predicted *y*-value data set using the model, ideally resulting in residuals that are small. The sum of the absolute values of the vertical 'residuals' between points generated by the model and corresponding points in the data was then calculated.

Solve accurate and appropriate use of technology	Figure 7 shows an excerpt from the residual analysis for both quadratic Model (A) and quadratic Model (B). An Excel spreadsheet was used to make the comparison and determine if either model was feasible. The full analysis and formulas used for the spreadsheet can be found in Appendix 1 and Appendix 2.

x-coordinate	*y*- coordinate	model A predicted y-coordinate	Absolute residual (model A)	model B predicated *y* - coordinate	Absolute residual (model B)
0	270	270	0	255.48	14.52
4	244.5	254.0079	9.50790123	239.8888	4.6112
7.5	230	240.61111	10.6111111	226.8345	3.1655
10.5	214.5	229.57111	15.0711111	216.0819	1.5819
13.5	200.5	218.94	18.44	205.7325	5.2325
17.5	186.5	205.40123	18.9012346	192.5605	6.0605
…	…				
173	237.5	242.49086	4.9908642	236.0694	1.4306
175	248	250.12346	2.12345679	243.685	4.315
177	260.5	257.93778	2.56222222	251.4798	9.0202
180	270	270	0	263.508	6.492
Total			**328.424444**		**198.0667**

Figure 7: Absolute residual analysis for analytic quadratic model (A) and Excel parabolic model (B) (excerpt).

	Using a similar procedure, the residual analysis for the Excel quartic polynomial model was found to be 552.17 (see Appendix 3).
Evaluate and verify evaluation of the reasonableness of solutions by considering the results, assumptions and observations	Neither a quadratic or a quartic model are valid models to represent how the chain hangs. The total variation is less using the technologically generated quadratic model (Model B), however the model required further refinement.
	Refining the model
Formulate accurate documentation of relevant observations; documentation of appropriate assumptions	Using Desmos software, it was observed that even-powered polynomials of the form $y = x^{2n}$ $(n \geq 1, n \in Z)$ always produced U-shaped sketches (see Appendix 4). An analytic procedure was used to generate the quartic function of the form $a(x-b)^{2n}+c$ for the hanging chain $(n = 2)$. The parameter values a, b and c move the sketch to the right and left, up and down and stretch or compress the sketch, but the shape remains U-

shaped therefore it is an appropriate assumption to use this type of function to model a chain.

Known coordinates:

Left fixation point $(0, 270)$

Right fixation point $(180, 270)$

Turning point $(\frac{180}{2}, 86) = (90, 86)$

Basic quartic function:

$y = x^4$ Turning point $(0, 0)$

Transformed function

$y = a(x + b)^4 + c$ (3)

The turning point of the basic function is $(0, 0)$, while the turning point of the transformed function is $(90, 86)$. The curve has been translated in the positive y-direction by 86mm, and translated in the positive x-direction by 90mm, producing the ‘c’ and ‘b’ parameter values of 86 and -90 respectively. Substituting these values into equation (3) gives:

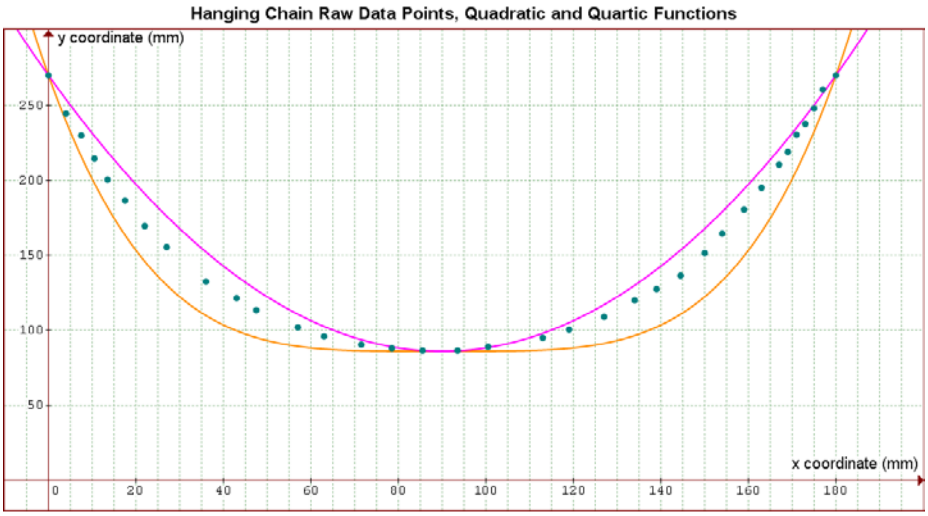
$y = a(x - 90)^4 + 86$ (4)

Communication
coherent and concise organisation of the response

Substituting $(0, 270)$ into equation (4) and using a similar procedure to the parabolic model on page 10 produced the following quartic function:

$y = \frac{23}{8201250}(x - 90)^4 + 86$ Model (C)

Figure 8 below shows the graph of model (C), the quadratic model (A) and the raw data values.



Solve
accurate and appropriate use of technology

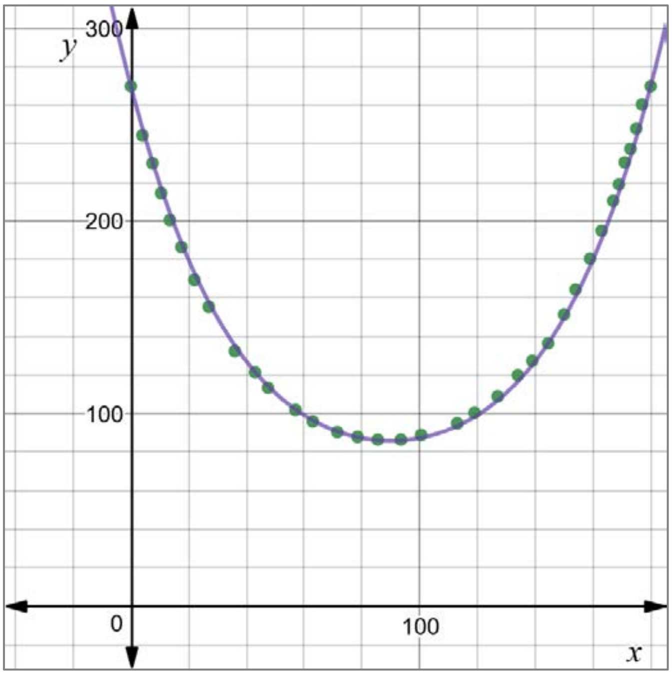
Figure 8: Graph showing the (x, y) position data of a freely hanging chain, the analytical quadratic model (A) (pink) and the analytical quartic model (C) (orange).

Evaluate and verify
documentation of relevant strengths and limitations of the model

Formulate
accurate documentation of relevant observations; documentation of appropriate assumptions;

Figure 8 shows that while the quadratic function (model (A)) is too narrow, the quartic function (model (C)) is too wide. It was assumed a combination of the two functions would give a better fit. New parameter values were required as clearly the sum of the quadratic and quartic functions would produce a result that was incorrect (e.g. adding the y-values for a given x-value for both functions would produce a coordinate which clearly does

accurate translation of all aspects of the problem **Solve** accurate use of complex procedures; discerning application of mathematical concepts and techniques relevant to the task **Communicate** correct use of appropriate technical vocabulary, procedural vocabulary, and conventions to develop the response	not lie on the chain). Known coordinates: Left fixation point $(0, 270)$ Point along the chain $(36.0, 132.5)$ Combination function 1: $y = a \times \frac{46}{2025}(x-90)^2 + b \times \frac{23}{8201250}(x-90)^4 + 86$ Setting up a simpler form of the function to solve for the parameter values A and B: $y = \mathrm{A}(x-90)^2 + \mathrm{B}(x-90)^4 + 86 \quad (5)$ Substituting the points $(0, 270)$ into equation (5): $270 = A(0-90)^2 + \mathrm{B}(0-90)^4 + 86$ $270 - 86 = A(-90)^2 + \mathrm{B}(-90)^4$ $184 = 8100\mathrm{A} + 90^4\mathrm{B}$ $184 = 8100(A + 8100B)$ $\frac{184}{8100} - 8100B = A \quad (6)$ Substitute $(36.0, 132.5)$ into equation (5) $132.5 = A(36-90)^2 + B(36-90)^4 + 86$ $132.5 = A(-54)^2 + \mathrm{B}(-54)^4 + 86$ $132.5 - 86 = 54^2\ (\mathrm{A} + 54^2\mathrm{B})$ $\frac{46.5}{2916} - 2916B = A \quad (7)$ Substitute equation (6) into equation (7) $\frac{184}{8100} - 8100B = \frac{46.5}{2916} - 2916B$ $-5184B = -0.0067695$ $B = 0.0000013059$ (to 5 significant figures) Substitute B into equation (6) $A = \frac{184}{8100} - 8100 \times 0.0000013059$ $A = 0.012138$ Combination function 1: $y = 0.012139 \times (x-90)^2 + 0.0000013059(x-90)^4 + 86$ Model (D) Combination function 2: The procedure was repeated using a different subset of two points $(0{,}270)$ and $(144.5, 136.5)$ and produced the following model: $y = 0.013693(x-90)^2 + 0.0000011139(x-90)^4 + 86$ Model (E)

Evaluate and verify justification of decisions made using mathematical reasoning; documentation of relevant strengths and limitations of the model	The sum of the absolute residuals for model (D) is 105.75 and for model (E) is 82.94. The most feasible analytical function to use for modelling the chain is model (E) which resulted in the lowest deviation of actual values from the y values generated using the model (as indicated by the smallest absolute residual sum). It should be noted however, that another subset of two points could be used to produce a different model; and consequently a different residual analysis could be considered. The graph of the model (E) is given below in Figure 9. The points are also plotted.  Figure 9: Graph showing the *(x, y)* position data of a freely hanging chain and model (E) — the sum of even powered polynomials.
Formulate accurate translation of all aspects of the problem by identifying techniques **Solve** accurate and appropriate use of technology	Technology was used to generate an alternative model. Using Desmos, the original data values (x_1, y_1) were input in a table and the function type was given, as shown below: $y \sim (a(x_1 - b)^2 + c(x_1 - d)^4 + f$ (see Figure 10). Note: the tilde (~) notation is used in Desmos to signify a regression analysis; the parameter values are adjusted to fit the data as closely as possible.

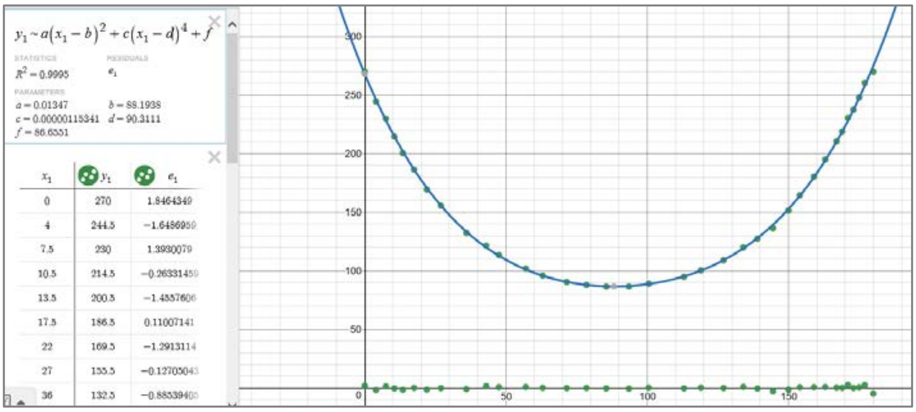


Figure 10: Graph showing the *(x, y)* position data of a freely hanging chain and the technology-generated model using the given function type.

Solve
accurate and appropriate use of technology

The following parameter values were generated:

$y = 0.01347(x - 88.1938)^2 + 0.00000115341(x - 90.3111)^4 + 86.6551$
Model (F) $(R^2 = 0.9995)$

Evaluate and verify
documentation of relevant strengths and limitations of the solution and/or model

The results for this model, and model (E) are very comparable. However, using Excel, the sum of the absolute residuals for the Desmos model (F) was 32.15077.

The most feasible model to use to represent how the chain hangs is therefore Model (F):

$y = 0.01347(x - 88.1938)^2 + 0.00000115341(x - 90.3111)^4 + 86.6551$
where $0 \leq x \leq 270$.

Communicate
coherent and concise organisation of the response including a conclusion

Conclusion

The observation that models of the form $y = a(x - b)^{2n}$ were always U-shaped in appearance provided the most successful approach to develop an accurate model for the shape of the chain. The model developed using the sum of even degree polynomial expressions was the most feasible. The summation of the quartic and quadratic models enabled the flatness of the quartic model and the narrowness of the quadratic model to be adjusted to more accurately model the hanging chain. A residual analysis provided better justification for the choice of the final model for the hanging chain. The polynomial models that were developed, produced residual results that were far greater than the summation model. The summation model that was generated using technology was the most valid. Further analysis could involve investigating the sum of a sixth-degree, fourth-degree polynomial and second-degree polynomial, or any combination of these.

Appendix 1

A	B	C	D	E	F
x coordinate	y coordinate	Analytic model (model A)	Absolute residual (model A)	Analytic model (model B)	Absolute residual (model B)
0	270	270	0	255.48	14.52
4	244.5	254.0079	9.5079012	239.8888	4.6112
7.5	230	240.61111	10.611111	226.8345	3.1655
10.5	214.5	229.57111	15.071111	216.0819	1.5819
13.5	200.5	218.94	18.44	205.7325	5.2325
17.5	186.5	205.40123	18.901235	192.5605	6.0605
22	169.5	191.03901	21.539012	178.5988	9.0988
27	155.5	176.16	20.66	164.1498	8.6498
36	132.5	152.24	19.74	140.964	8.464
43	121.5	136.17975	14.679753	125.4394	3.9394
47.5	113.5	127.03086	13.530864	116.6185	3.1185
57	102	110.73778	8.7377778	100.9758	1.0242
63	96	102.56	6.56	93.1794	2.8206
71.5	90.5	93.774568	3.2745679	84.8953	5.6047
78.5	88	89.004198	1.0041975	80.5035	7.4965
85.5	86.5	86.46	0.04	78.3069	8.1931
93.5	86.5	86.278272	0.2217284	78.4845	8.0155
100.5	89	88.504444	0.4955556	80.9919	8.0081
113	95	98.01679	3.0167901	90.9294	4.0706
119	100.5	105.1042	4.6041975	98.1858	2.3142
127	109	117.09827	8.0982716	110.3698	1.3698
134	120	129.97827	9.9782716	123.3828	3.3828
139	127.5	140.54123	13.041235	134.0218	6.5218
144.5	136.5	153.47235	16.972346	147.0183	10.5183
150	151.5	167.77778	16.277778	161.37	9.87
154	164.5	179.04494	14.544938	172.6588	8.1588
159	180.5	194.15111	13.651111	187.7778	7.2778
163	195	207.05383	12.053827	200.6794	5.6794
167	210.5	220.68346	10.183457	214.2978	3.7978
169	219	227.77086	8.7708642	221.3758	2.3758
171	230.5	235.04	4.54	228.633	1.867
173	237.5	242.49086	4.9908642	236.0694	1.4306
175	248	250.12346	2.1234568	243.685	4.315
177	260.5	257.93778	2.5622222	251.4798	9.0202
180	270	270	0	263.508	6.492
Total			328.42444		198.0667

Appendix 2

x coordinate	y coordinate	Analytic model (model A)	Absolute residual (model A)	Analytic model (model B)	Absolute residual (model B)
0	270	=46/2025*(A2-90)^2+8	=ABS(C2-B2)	=0.0224*A2^2-3.9874*A	=ABS(E2-B2)
4	244.5	=46/2025*(A3-90)^2+8	=ABS(C3-B3)	=0.0224*A3^2-3.9874*A	=ABS(E3-B3)
7.5	230	=46/2025*(A4-90)^2+8	=ABS(C4-B4)	=0.0224*A4^2-3.9874*A	=ABS(E4-B4)
10.5	214.5	=46/2025*(A5-90)^2+8	=ABS(C5-B5)	=0.0224*A5^2-3.9874*A	=ABS(E5-B5)
13.5	200.5	=46/2025*(A6-90)^2+8	=ABS(C6-B6)	=0.0224*A6^2-3.9874*A	=ABS(E6-B6)
17.5	186.5	=46/2025*(A7-90)^2+8	=ABS(C7-B7)	=0.0224*A7^2-3.9874*A	=ABS(E7-B7)
22	169.5	=46/2025*(A8-90)^2+8	=ABS(C8-B8)	=0.0224*A8^2-3.9874*A	=ABS(E8-B8)
27	155.5	=46/2025*(A9-90)^2+8	=ABS(C9-B9)	=0.0224*A9^2-3.9874*A	=ABS(E9-B9)
36	132.5	=46/2025*(A10-90)^2+	=ABS(C10-B10)	=0.0224*A10^2-3.9874	=ABS(E10-B10)
43	121.5	=46/2025*(A11-90)^2+	=ABS(C11-B11)	=0.0224*A11^2-3.9874	=ABS(E11-B11)
47.5	113.5	=46/2025*(A12-90)^2+	=ABS(C12-B12)	=0.0224*A12^2-3.9874	=ABS(E12-B12)
57	102	=46/2025*(A13-90)^2+	=ABS(C13-B13)	=0.0224*A13^2-3.9874	=ABS(E13-B13)
63	96	=46/2025*(A14-90)^2+	=ABS(C14-B14)	=0.0224*A14^2-3.9874	=ABS(E14-B14)
71.5	90.5	=46/2025*(A15-90)^2+	=ABS(C15-B15)	=0.0224*A15^2-3.9874	=ABS(E15-B15)
78.5	88	=46/2025*(A16-90)^2+	=ABS(C16-B16)	=0.0224*A16^2-3.9874	=ABS(E16-B16)
85.5	86.5	=46/2025*(A17-90)^2+	=ABS(C17-B17)	=0.0224*A17^2-3.9874	=ABS(E17-B17)
93.5	86.5	=46/2025*(A18-90)^2+	=ABS(C18-B18)	=0.0224*A18^2-3.9874	=ABS(E18-B18)
100.5	89	=46/2025*(A19-90)^2+	=ABS(C19-B19)	=0.0224*A19^2-3.9874	=ABS(E19-B19)
113	95	=46/2025*(A20-90)^2+	=ABS(C20-B20)	=0.0224*A20^2-3.9874	=ABS(E20-B20)
119	100.5	=46/2025*(A21-90)^2+	=ABS(C21-B21)	=0.0224*A21^2-3.9874	=ABS(E21-B21)
127	109	=46/2025*(A22-90)^2+	=ABS(C22-B22)	=0.0224*A22^2-3.9874	=ABS(E22-B22)
134	120	=46/2025*(A23-90)^2+	=ABS(C23-B23)	=0.0224*A23^2-3.9874	=ABS(E23-B23)
139	127.5	=46/2025*(A24-90)^2+	=ABS(C24-B24)	=0.0224*A24^2-3.9874	=ABS(E24-B24)
144.5	136.5	=46/2025*(A25-90)^2+	=ABS(C25-B25)	=0.0224*A25^2-3.9874	=ABS(E25-B25)
150	151.5	=46/2025*(A26-90)^2+	=ABS(C26-B26)	=0.0224*A26^2-3.9874	=ABS(E26-B26)
154	164.5	=46/2025*(A27-90)^2+	=ABS(C27-B27)	=0.0224*A27^2-3.9874	=ABS(E27-B27)
159	180.5	=46/2025*(A28-90)^2+	=ABS(C28-B28)	=0.0224*A28^2-3.9874	=ABS(E28-B28)
163	195	=46/2025*(A29-90)^2+	=ABS(C29-B29)	=0.0224*A29^2-3.9874	=ABS(E29-B29)
167	210.5	=46/2025*(A30-90)^2+	=ABS(C30-B30)	=0.0224*A30^2-3.9874	=ABS(E30-B30)
169	219	=46/2025*(A31-90)^2+	=ABS(C31-B31)	=0.0224*A31^2-3.9874	=ABS(E31-B31)
171	230.5	=46/2025*(A32-90)^2+	=ABS(C32-B32)	=0.0224*A32^2-3.9874	=ABS(E32-B32)
173	237.5	=46/2025*(A33-90)^2+	=ABS(C33-B33)	=0.0224*A33^2-3.9874	=ABS(E33-B33)
175	248	=46/2025*(A34-90)^2+	=ABS(C34-B34)	=0.0224*A34^2-3.9874	=ABS(E34-B34)
177	260.5	=46/2025*(A35-90)^2+	=ABS(C35-B35)	=0.0224*A35^2-3.9874	=ABS(E35-B35)
180	270	=46/2025*(A36-90)^2+	=ABS(C36-B36)	=0.0224*A36^2-3.9874	=ABS(E36-B36)
Total			=SUM(D2:D36)	=0.0224*A37^2-3.9874	=SUM(F2:F36)

Appendix 3

x-coordinate	*y*-coordinate	Technology-generated quartic regression model	Absolute residual
0	270	268.15	1.85
4	244.5	246.145856	1.645856
7.5	230	228.6090391	1.390960938
10.5	214.5	214.7754301	0.275430062
13.5	200.5	201.9852901	1.485290062
17.5	186.5	186.4566641	0.043335938
22	169.5	170.922056	1.422056
27	155.5	155.859241	0.359241
36	132.5	133.882816	1.382816
43	121.5	120.716201	0.783799
47.5	113.5	113.8045391	0.304539062
57	102	102.598801	0.598801

63	96	97.536361	1.536361
71.5	90.5	92.55857506	2.058575062
78.5	88	90.08740006	2.087400062
85.5	86.5	88.86303006	2.363030063
93.5	86.5	88.80301006	2.303010062
100.5	89	89.82532506	0.825325062
113	95	94.095761	0.904239
119	100.5	97.332521	3.167479
127	109	103.022441	5.977559
134	120	109.494536	10.505464
144.5	136.5	122.3995151	14.10048494
150	151.5	131.005	20.495
154	164.5	138.199256	26.300744
159	180.5	148.435561	32.064439
163	195	157.725161	37.274839
167	210.5	168.094121	42.405879
169	219	173.714321	45.285679
171	230.5	179.642281	50.857719
173	237.5	185.891441	51.608559
175	248	192.475625	55.524375
177	260.5	199.409041	61.090959
180	270	210.496	59.504
Total			**552.1696033**

Appendix 4