Mathematical Methods 2019 v1.2

Units 1 and 2 sample marking scheme

September 2018

Examination — short response

This sample has been compiled by the QCAA to model one possible approach to allocating marks in an examination. It matches the examination mark allocations as specified in the syllabus (~ 60% simple familiar, ~ 20% complex familiar and ~ 20% complex unfamiliar) and ensures that all the objectives are assessed.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from Units 1 and 2
2. comprehend mathematical concepts and techniques drawn from Units 1 and 2
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from Units 1 and 2.

Task

See the sample assessment instrument for Units 1 and 2: Examination — short response (available on the QCAA Portal).

Sample marking scheme

Criterion	Marks allocated	Result
Foundational knowledge and problem-solving Assessment objectives 1, 2, 3, 4, 5 and 6	—	—
Total	—	—

The annotations are written descriptions of the expected response for each question and are related to the assessment objectives.

Note: ✓ = $\frac{1}{2}$ mark

Marking scheme

Paper 1 (technology-free)

Question 1 (4 marks) SF

1a. recognise feature

a. Amplitude = 3✓

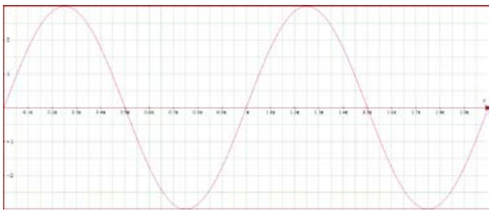
1b. select rule and perform calculation

b. Period $= \frac{2\pi}{2} = \pi$✓✓

1c. recall shape; recognise domain and amplitude

communicate information graphically (label axes accurately)

c.



✓✓✓

✓✓

Question 2 (4 marks) SF

2. select and use facts and procedures to determine value for each term on LHS

Changing each term to index form:

$\log_3 27 = x \to 27 = 3^x \therefore x = 3$✓

$\log_8 \frac{1}{8} = x \to \frac{1}{8} = 8^x \therefore x = -1$✓✓

$\log_{16} 4 = x \to 4 = 16^x = 4^{2x} \to 1 = 2x \therefore x = \frac{1}{2}$✓✓

LHS $= 3 - 1 - \frac{1}{2} = \frac{3}{2}$

use rule to generate an equation in x

$\frac{3}{2} = \log_4 x$✓

use index rules to determine x

$x = 4^{\frac{3}{2}} = (2^2)^{\frac{3}{2}}$✓

$x = 2^3 = 8$✓

3a.

recall and use rule for differentiating polynomial and power functions

recall derivative of a constant is zero

3b.

recognise composite function

use chain rule

substitute the derivatives

3c.

select and use product rule

select and use chain rule

recall and use rules for differentiating polynomial and power functions

3d.

select and use quotient rule

recall and use rules for differentiating polynomial functions

Question 3 (11 marks) SF

✓✓✓✓✓

a. $8x^3 - 6x - \frac{5}{x^2}$

✓

b. Use the chain rule

Let $u = (x^3 + x)$✓✓

$\frac{dy}{dx} = 5(u)^4 \times u'$✓✓

$\frac{dy}{dx} = 5(x^3 + x)^4 \times (3x^2 + 1)$✓✓

c. Use the chain rule within product rule

✓✓

✓ ✓✓ ✓

$\frac{dy}{dx} = (2x^2 + 3)\frac{1}{2\sqrt{x-1}} + \sqrt{x-1}(4x)$

d. Use quotient rule

✓✓ ✓ ✓

$\frac{(x^3-3)\times-1-((1-x)3x^2)}{(x^3-3)^2}$

Question 4 (4 marks) CF

$f'(x) = 3ax^2 - 12x$✓✓

At point $P(x = 1)$ the gradient is 3 ∴

✓✓ ✓✓

$3 = 3a \times 1^2 - 12 \times 1$

$a = 5$✓✓

4.

translate information to generate derivative

comprehend and use information to create an equation in a

determine solution for a

5a.

select and use rule

perform calculation

5b.

select rule

put result into effect

perform calculation

Question 5 (5 marks) SF

a. $\sum p_i = 1$

$0.3 + a + 2a + 0.1 = 1$✓✓

$a = 0.2$✓✓

b.

✓✓

$E(X) = 0.3 \times 0 + 2 \times 0.2 + 5 \times 2 \times 0.2 + 9 \times 0.1$✓✓

$E(X) = 3.3$ ✓✓

Question 6 (8 marks) SF

a. $2\sin(x) = 1$✓

$\sin(x) = \frac{1}{2}$✓

✓ ✓✓

$x = \frac{\pi}{6}, \quad \frac{5\pi}{6}$

b. $2\cos^2(x) - \cos(x) = 0$

$\cos(x)\,(2\cos(x) - 1) = 0$✓

$\therefore 2\cos(x) - 1 = 0$✓

$\cos(x) = \frac{1}{2}$✓

✓ ✓✓

$x = \frac{\pi}{3}, \quad \frac{5\pi}{3}$

or

$\cos(x) = 0$ ✓

✓ ✓

$x = \frac{\pi}{2}, \quad \frac{3\pi}{2}$

✓✓

6a.

recall and use procedure for rearranging an equation

recall exact value for $\sin(x)$ and integer multiples of $\frac{\pi}{6}$

6b.

recall and use procedure for solving a quadratic equation (factorising or quadratic formula)

recall and use procedure for rearranging an equation

recall exact value for $\cos(x)$ and integer multiples of $\frac{\pi}{6}$

communicate solutions appropriately using given domain (radians)

Question 7 (6 marks) SF

Solve for parameters in function:

$y = A\sin(B(x + C)) + D$ (or cosine function)✓

Amplitude (A) = $\frac{8}{2} = 4$ ✓

Period = $\frac{2\pi}{B} = \pi \therefore B = \frac{2\pi}{\pi} = 2$ ✓✓

$C = 0$ (curve has not been translated horizontally for sine function)

(Note: for cosine function $C = -\frac{\pi}{4}$) ✓✓

$D = -3$ (curve has been translated vertically downwards 3 units)✓✓

✓✓

$y = 4\sin(2x) - 3$

✓✓

(Note: cosine function model is $y = 4\cos(2\left(x - \frac{\pi}{4}\right)) - 3$ or equivalent.)

7.

comprehend periodic model is appropriate

recall procedures and put into effect to determine parameter values

justify all procedures and decisions

communicate equation of graph

8a.

recall use of tree diagram to organise and visualise the different possible outcomes of dependent events

8b.

recall and use procedure for multiplying along the branches

8c.

recognise and use conditional probability rule

recall and use procedure for

- multiplying along the branches
- adding the probabilities
- determining the probability

Question 8 (7 marks) SF

a. First branch $P(\text{Alice wins}) = \frac{2}{9}$✓

Second branch (top) $P(\text{Alice wins}) = \frac{2}{7}$✓

Second branch (bottom) $P(\text{Alice wins}) = \frac{1}{3}$✓

b. $P(\text{Raoul wins the first and Alice wins the second})$

$= \frac{7}{9} \times \frac{2}{7}$✓

$= \frac{2}{9}$✓

c. $P(\text{Raoul wins both games/he wins at least one game})$✓

$= \frac{P(\text{Raoul wins both games} \cap \text{Raoul wins at least one game})}{P(\text{Raoul wins at least one game})}$

✓✓

$= \frac{\frac{7}{9} \times \frac{5}{7}}{\left(\frac{7}{9} \times \frac{5}{7}\right) + \left(\frac{7}{9} \times \frac{2}{7}\right) + \left(\frac{2}{9} \times \frac{2}{3}\right)}$

✓✓ ✓ ✓✓

$= \frac{15}{25} = \frac{3}{5}$✓

Question 9 (6 marks) CU

Stationary point $y' = 0$ ✓

$y' = 4x^3 + 16x - 20 = 4(x^3 + 4x - 5)$

Using trial and error, $(x - 1)$ is a factor of y'✓

Use factor theorem:

$x^3 + 4x - 5 = (x - 1)(x^2 + x + 5)$✓✓

The discriminant of $(x^2 + x + 5)$ is negative therefore no real roots.

✓✓

So $(x - 1)$ is the only factor and $x = 1$ the only stationary point.

To determine the nature of the point $x = 1$

$y'(0) = -20$✓

$y'(2) = 44$✓

Slope is negative to the left and positive to the right

∴ stationary point at $x = 1$ is a minimum.✓✓

✓✓

9.

recall use of derivative to determine stationary point

comprehend

- use of given information to determine factor of derivative
- use factor theorem to factorise derivative
- use of discriminant

use first derivative rule to determine the nature of the stationary point

justify all procedures and decisions

Paper 2 (technology-active)

Question 1 (2 marks) SF

1.

select and use rule for independent events

determine solution for k

$$P(C \cap D) = P(C) \times P(D)$$
$$= 2k \times 3k^2 \checkmark$$
$$= 6k^3 \checkmark$$
$$6k^3 = 0.162 \checkmark$$
$$k^3 = 0.027$$
$$k = 0.3 \checkmark$$

Question 2 (9 marks) 3 SF, 6 CU

2a.

recall and use rules

communicate clearly (as cost function is given)

a. $\text{Speed} = \frac{\text{distance}}{\text{time}} \checkmark$

$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{k}{v}\ (i) \checkmark$

Cost = $(2 + 0.001v^3) \times t \checkmark$

Substitute (i)

Cost = $(2 + 0.001v^3) \times \frac{k}{v} \checkmark$

Cost = $2kv^{-1} + 0.001kv^2$ ✓✓

2b.

comprehend that use of derivative is required

justify procedures and decisions

determine stationary value using

- factorisation method
- index law/ technology

use first derivative test to verify result is a minimum

justify procedures and decisions

b. $C' = -2kv^{-2} + 0.002kv$ ✓✓

Stationary point (minimum cost) $C' = 0$ ✓

$$0 = k\left(\frac{-2}{v^2} + 0.002v\right) \checkmark$$

$\frac{2}{v^2} = 0.002v$ ✓ (Note: equation is independent of k.) ✓

$$1000 = v^3 \checkmark$$

$$v = 10 \checkmark$$

Show that $v = 10$ is a minimum.

$$C'(9) = \frac{-2k}{81} + 0.018k$$
$$= -0.0067k \checkmark$$

$$C'(11) = \frac{-2k}{121} + 0.022k$$
$$= 0.00547k \checkmark$$

Gradient is negative to the left, and positive to the right ∴ stationary point is a minimum. ✓✓

3.

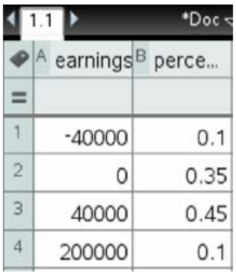
select appropriate method of solution

use technology appropriately

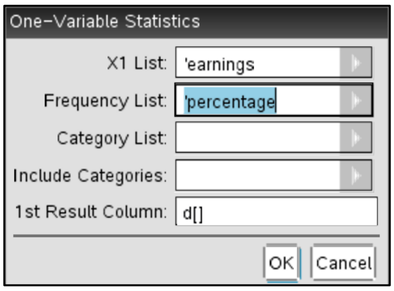
Question 3 (4 marks) SF

Technological solution (see analytical solution below)

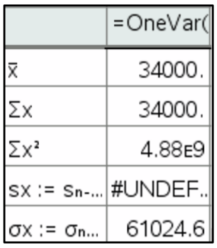
Nspire ✓✓



spreadsheet ✓✓



one variable statistics ✓✓



mean and standard deviation ✓✓

Analytical solution

use given formula

$X =$ possible profit in thousands

x	$P(x)$	$xP(x)$ ✓✓	$P(x) \times (x - \mu)^2$ ✓✓
–40	0.1	–4	547.6
0	0.35	0	404.6
40	0.45	18	16.2
200	0.1	20	2755.6
		$\sum xP(x) = 34$	$\sum P(x) \times (x - \mu)^2 = 3724$

Expected value = $E(x) = \mu = 34$

Variance = 3724

Standard deviation = $\sqrt{3724} = 61.0246$

therefore

Expected profit = $34 000 ✓✓

Standard deviation = $61 024.60 ✓✓

Question 4 (5 marks) SF

4a.

recall and use rule to determine the common ratio

4b.

use rule to determine the nth term in a geometric sequence

4c.

comprehend sum to infinity is required

use rule to determine the sum to infinity

a. $r = \frac{u_2}{u_1}$

$= \frac{-625}{2500}$✓

$= -0.25$✓

b. $t_{10} = 2500 \times (-0.25)^9$✓✓

$= .009\ 536$✓✓

c. $S_\infty = \frac{2500}{1+0.25}$✓✓

$S_\infty = 2000$✓✓

Question 5 (6 marks) CF

5.

comprehend value of producing a sketch with local maxima and minima values labelled

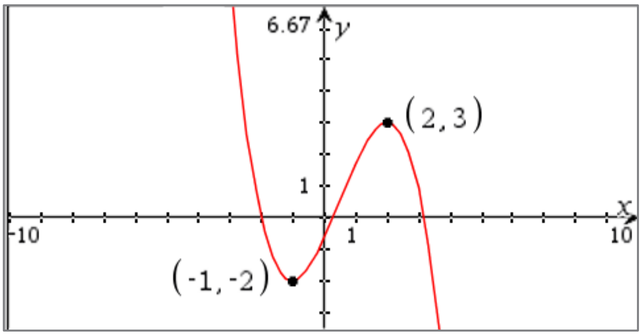
recall $g(x) = 0$ at the x-intercepts

recall k value moves curve vertically so y-coordinates of stationary points are used

justify procedures using knowledge of transformations

recognise the solution set for k has two intervals

communicate solution



✓✓

$g(x) = 0$ represents points where the function cuts the x-axis ✓✓

✓✓

Moving the graph down 3 units will result in two x-intercepts

$\therefore k < -3$ ✓✓

similarly moving the graph up 2 units will result in two x-intercepts

$\therefore k > 2$ ✓✓

✓✓

6.

comprehend use of derivative is required

use rules to determine derivative

translate stationary value to mathematical equations

recall and use method to determine a simultaneous solution

recognise quadratic form of equation and solve for unknown

describe and communicate method used for solution clearly

evaluate the reasonableness of the solution

Question 6 (9 marks) CF

Expand (or use product rule) and differentiate

$f(x) = x^3 + px^2 + qx + pq$ ✓

$f'(x) = 3x^2 + 2px + q$ ✓

Given stationary value at point $(1, -2)$

Substitute into $f(x)$

$-2 = 1 + p + q + pq \ (i)$ ✓

$f'(1) = 0$✓

$0 = 3 + 2p + q \ (ii)$✓

Rearrange (ii)

$q = -2p - 3$

Substitute into (i) ✓

$-2 = 1 + p - 2p - 3 + p(-2p - 3)$ ✓

$-2 = -2p^2 - 4p - 2$

$0 = -2p^2 - 4p$ ✓

Solve quadratic equation (using factorisation or otherwise)

$0 = -2p(p + 2)$✓✓

$p = 0, -2$ ✓✓

$\therefore q = -3, 1$ ✓✓

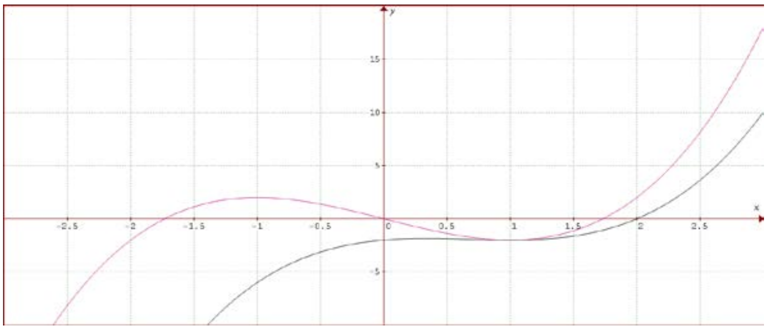
So $f(x) = x^3 - 3x$

or $f(x) = x^3 - 2x^2 + x - 2$

✓✓

Each function is graphed and stationary point at

(1, –2) identified ✓✓



7.

determine the periodic model (using technology or otherwise)

Question 7 (7 marks) CU

Using technology to determine a periodic regression model ✓✓

containing the points: ✓✓

(0, 10)

(24, 10)

(6, 14)

(18, 6)

or use analytical skills to determine a model

$y = 4\sin\left(\frac{\pi}{12}t\right) + 10$ ✓✓

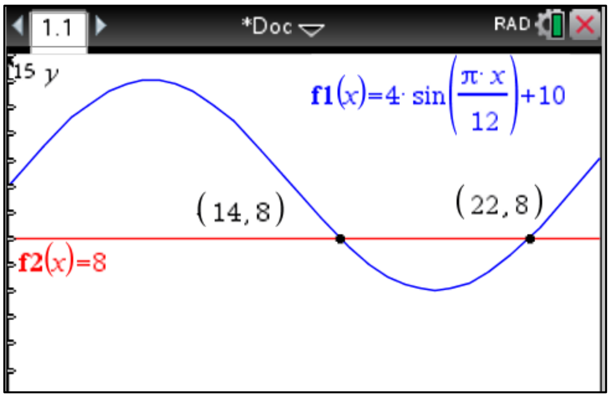
(Note: points (6, 6) and (18, 14) may be used to generate the model.)

$y = -4\sin\left(\frac{\pi}{12}t\right) + 10$

Determine points of intersection with line $y = 8$

translate information into a mathematically workable format and identify the mathematical procedure required

- points of intersection between periodic model and line $y = 8$
- time difference between points of intersection
- use of probability rule
- explain all mathematical reasoning



✓

(14, 8) and (22, 8) ✓✓

Amount of time temperature is 8°C or lower

$22 - 14 = 8$ ✓✓

	$\therefore P \leq 8 = \frac{8}{24}$ ✓ $P(\text{Temperature} \leq 8°C) \approx 33.3\%$ (or 0.333) ✓✓	