Mathematical Methods 2019 v1.2

IA2 sample marking scheme

March 2024

Examination (15%)

This sample has been compiled by the QCAA to model one possible approach to allocating marks in an examination. It matches the examination mark allocations as specified in the syllabus (~ 60% simple familiar, ~ 20% complex familiar and ~ 20% complex unfamiliar) and ensures that a balance of the objectives are assessed.

Assessment objectives

This assessment instrument is used to determine student achievement in the following objectives:

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 3 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 3 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 3 topics.

Instrument-specific marking guide (ISMG)

Criterion: Foundational knowledge and problem-solving

Assessment objectives

1. select, recall and use facts, rules, definitions and procedures drawn from all Unit 3 topics
2. comprehend mathematical concepts and techniques drawn from all Unit 3 topics
3. communicate using mathematical, statistical and everyday language and conventions
4. evaluate the reasonableness of solutions
5. justify procedures and decisions by explaining mathematical reasoning
6. solve problems by applying mathematical concepts and techniques drawn from all Unit 3 topics

The student work has the following characteristics:	Cut-off	Marks
• consistently correct selection, recall and use of facts, rules, definitions and procedures; authoritative and accurate command of mathematical concepts and techniques; astute evaluation of the reasonableness of solutions and use of mathematical reasoning to correctly justify procedures and decisions; and fluent application of mathematical concepts and techniques to solve problems in a comprehensive range of simple familiar, complex familiar and complex unfamiliar situations.	> 93%	15
	> 87%	14
• correct selection, recall and use of facts, rules, definitions and procedures; comprehension and clear communication of mathematical concepts and techniques; considered evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions; and proficient application of mathematical concepts and techniques to solve problems in simple familiar, complex familiar and complex unfamiliar situations.	> 80%	13
	> 73%	12
• thorough selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of solutions and use of mathematical reasoning to justify procedures and decisions; and application of mathematical concepts and techniques to solve problems in simple familiar and complex familiar situations.	> 67%	11
	> 60%	10
• selection, recall and use of facts, rules, definitions and procedures; comprehension and communication of mathematical concepts and techniques; evaluation of the reasonableness of some solutions using mathematical reasoning; and application of mathematical concepts and techniques to solve problems in simple familiar situations.	> 53%	9
	> 47%	8
• some selection, recall and use of facts, rules, definitions and procedures; basic comprehension and communication of mathematical concepts and techniques; inconsistent evaluation of the reasonableness of solutions using mathematical reasoning; and inconsistent application of mathematical concepts and techniques.	> 40%	7
	> 33%	6
• infrequent selection, recall and use of facts, rules, definitions and procedures; basic comprehension and communication of some mathematical concepts and techniques; some description of the reasonableness of solutions; and infrequent application of mathematical concepts and techniques.	> 27%	5
	> 20%	4
• isolated selection, recall and use of facts, rules, definitions and procedures; partial comprehension and communication of rudimentary mathematical concepts and	> 13%	3
	> 7%	2

techniques; superficial description of the reasonableness of solutions; and disjointed application of mathematical concepts and techniques.		
• isolated and inaccurate selection, recall and use of facts, rules, definitions and procedures; disjointed and unclear communication of mathematical concepts and techniques; and illogical description of the reasonableness of solutions.	> 0%	1
• does not satisfy any of the descriptors above.		0

Task

See IA2 sample assessment instrument: Examination (15%) (available on the QCAA Portal).

Sample marking scheme

Criterion	Marks allocated	Provisional marks
Foundational knowledge and problem-solving Assessment objectives 1, 2, 3, 4, 5, 6	15	—
Total	**15**	**—**

The annotations are written descriptions of the expected response for each question and are related to the assessment objectives.

Note: ✓ = $\frac{1}{2}$ mark

Marking scheme — Paper 1 (technology-free)

Question 1 (SF 7 marks)

1a.
select and use:
- logarithm of a power rule
- logarithm of a division rule OR logarithm of 1 rule
- logarithm of a product rule
- facts to solve

a. $x = 2\log_6(3) + \log_6(4) \quad - \log_6(1)$
$x = \log_6(9) + \log_6(4) \quad - 0$
✓✓ ✓✓

$x = \log_6(36)$ ✓✓

$x = 2$ ✓✓

1b.
comprehend the information is in factorised form

select and use:
- null factor procedure
- rules to change from index form to log form to c solutions for x

b. $(e^x - 2)(e^x - 3) = 0$
Use null factor theorem: ✓✓
$e^x = 2$ and $e^x = 3$ ✓✓
$x = \ln 2$ and $x = \ln 3$ ✓✓

Question 2 (SF 13 marks)

2a.
select and use:
- derivative of e^x
- procedure for chain rule (recognise inner and outer function)
- derivative of a trigonometric function

a. $f(x) = e^x + \sin(2x)$

$f'(x) = e^x + 2\cos(2x)$
✓✓ ✓ ✓ ✓✓

2b.
select and use:
- procedure for chain rule to differentiate an exponential function and trigonometric function (recognise inner and outer function)

b. $f(x) = e^{\sin(x)}$
$f'(x) = e^{\sin(x)} \times \cos(x)$
✓✓✓✓ ✓✓

2c.
select and use:
- procedure for chain rule to differentiate a polynomial and a trigonometric function

c. $f(x) = \cos^3(x)$
$f'(x) = 3(\cos(x))^2 \times -\sin(x)$
✓✓✓✓ ✓✓
or $-3\cos^2(x)\sin(x)$

d. $f(x) = x + x\ln(x)$

$$f'(x) = 1 + \overbrace{x \times \frac{1}{x} + \ln(x) \times 1}^{\checkmark\checkmark}$$

$$\checkmark\checkmark \quad \checkmark\checkmark$$

$$f'(x) = 2 + \ln(x) \checkmark\checkmark$$

2d.
select and use:
- rule for derivative of a polynomial
- rule for differentiating a product
- rule for derivative of natural logarithm
- facts to simplify response

Question 3 (SF 7 marks)

3a.
select and use:
- rules for integrating a polynomial function
- procedure for calculating the value of a definite integral
- facts to determine the value of the definite integral

a. $\int_1^3 4x^2 dx$

$= \frac{4}{3}x^3|_1^3$ ✓✓

$= \frac{4}{3}(3^3 - 1^3)$ ✓

$= \frac{104}{3}$ ✓✓

b. $\int_0^2 6e^{2t} + t\, dt$

$= \frac{6e^{2t}}{2} + \frac{t^2}{2}|_0^2$

✓✓ ✓✓

$= \left(3e^4 + \frac{2^2}{2}\right) - \left(\frac{6e^0}{2} + \frac{0}{2}\right)$ ✓✓

$= 3e^4 + 2 - 3$ ✓

$= 3e^4 - 1$✓✓

3b.
select and use:
- rules for integrating polynomial function
- rules for integrating an exponential function
- procedure for calculating the value of a definite integral

recall rule $a^0 = 1$

determine value of the definite integral

Question 4 (CF 6 marks)

4.
identify $x'(t) = v(t)$

select and use procedure for differentiating a product to determine $v(t)$

translate information into mathematically workable format (determine $v(t) = 0$)

select and use procedures (factorising, null factor theorem)

recall log laws/exponential function asymptote to identify solution that is ‘not possible’
rearrange to generate trigonometric equation
recall common ratios to determine solution to trigonometric equation

Given $x(t) = e^t \sin(t)$
$x'(t) = v(t)$ ✓
$x'(t) = e^t \cos t + \sin t \times e^t$ ✓✓
Particle is at rest when $x'(t) = 0$ ✓✓

$0 = e^t \cos t + \sin t \times e^t$
$0 = e^t(\cos(t) + \sin(t))$ ✓

$\therefore e^t = 0$ *not possible* ✓✓

and $\cos(t) + \sin(t) = 0$ ✓
$\cos(t) = -\sin(t)$
$\tan(t) = -1$ ✓

Recall common ratios and CAST rule

$t = \frac{3\pi}{4}$ ✓ and $\frac{7\pi}{4}$ ✓

5. translate information into mathematically workable format (maximum point occurs when $f'(x) = 0$) select and use: • procedure for differentiating a quotient, differentiating $\ln(x)$ and polynomials to determine $f'(x)$ • procedures for rearranging the equation • rules for changing from log form to index form • substitution to determine y-coordinate of maximum point • rules for changing from log form to index form communicate the coordinates of the maximum point using appropriate terminology	## Question 5 (CF 6 marks) Given $f(x) = \frac{\ln(2x)}{x}, x > 0$ Maximum occurs when $f'(x) = 0$ ✓ Use quotient rule: ✓ $f'(x) = \frac{x \times \frac{2}{2x} - (\ln(2x) \times 1)}{x^2}$ ✓✓ Maximum point $f'(x) = 0$ $0 = \frac{1-\ln(2x)}{x^2}$ ✓ $0 = 1 - \ln(2x)$ ✓ $\ln(2x) = 1$ ✓ $2x = e$ $x = \frac{e}{2}$ ✓ $f\left(\frac{e}{2}\right) = \frac{\ln\left(2 \times \frac{e}{2}\right)}{\frac{e}{2}}$ ✓ $= \frac{2\ln e}{e}$ $= \frac{2}{e}$ ✓ Maximum point of the function $\left(\frac{e}{2}, \frac{2}{e}\right)$ ✓✓	
	## Question 6 (CF 5 marks) Given at time t = 3 population is 50: $P(3) = 50 = \frac{100}{1+e^{b-t}}$ ✓ $50(1 + e^{b-3}) = 100$ $1 + e^{b-3} = 2$ ✓ $e^{b-3} = 1$ $\therefore b = 3$ ✓ Using chain rule: $P'(t) = -100(1 + e^{3-t})^{-2} \times -e^{3-t}$ ✓✓ ✓✓ $P'(3) = \frac{-100 \times e^0}{(1 + e^0)^2}$ ✓ $P'(3) = \frac{100}{4} = 25$ ✓ The flu is spreading at a rate of 25 students/day on day 3.✓	**6.** identify critical elements: • $P(3) = 50$ • $P'(3)$ is required use algebraic skills to determine b use rules to: • determine an expression for $P'(t)$ select and use procedure to: • generate an equation using substitution • solve equation communicate findings

Marking scheme — Paper 2 (technology-active)

Question 7 (SF 9 marks)

7a. recognise qualitative features of the logarithmic graph

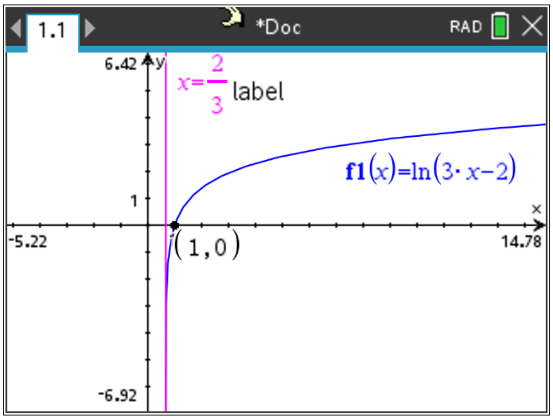
use appropriate notation (symbols) for domain

a. Asymptote $x = \frac{2}{3}$ ✓✓

x -intercept $(1,0)$ ✓✓

Sketch – general shape of curve✓✓

Axes etc labelled. ✓✓



7b. determine derivatives

b. $f'(x) = \frac{3}{3x-2}$ ✓✓

7c. determine derivatives

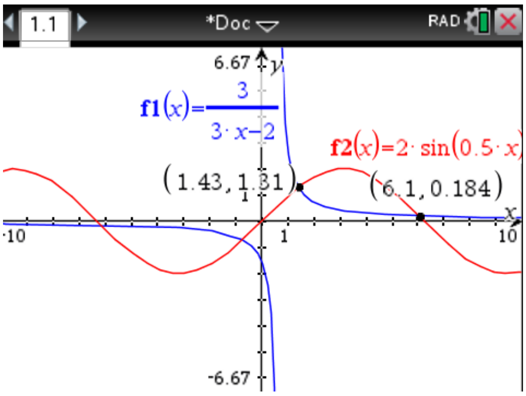
c. $g'(x) = 2\sin(0.5x)$ ✓✓

7d. Comprehend information to translate into a mathematically workable format

d. The gradients will be equal when:

$f'(x) = g'(x)$ ✓✓

Solve by graphing the derivatives and identifying points of intersection: ✓✓

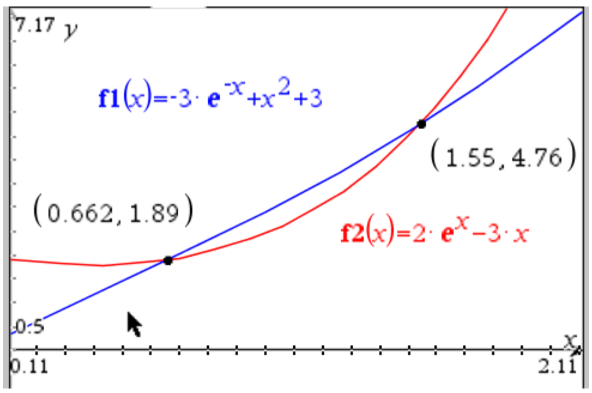


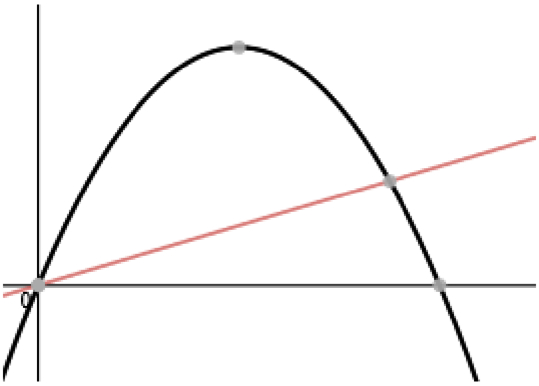
use an appropriate method to determine the values of x

Gradients are the same at $x = 1.43$ and $x = 6.1$ ✓✓

Question 8 (SF 8 marks)

a. Given $N = 4200$ when $t = 8, n(t) = Ae^{bt}$

8a. translate information into a mathematically workable format (substitute) determine A	$4200 = Ae^{0.55\times 8}$ ✓✓ $A = \dfrac{4200}{e^{0.55\times 8}}$ $A = 51.5648$ ✓✓	
8b. use substitution to generate equation for estimate comprehend '12 years after they were introduced' requires t value of 12 determine $n(12)$	b. Determine $n(12)$ $n(12) = 51.5648e^{0.55\times 12}$ ✓✓ $n(12) = 37905.036$ ✓ 12 years after they were introduced, the population will be approximately 37905.✓	
	c. $n(t) = 51.5648e^{0.55t}$ $n'(t)$ models the rate of change of the population ✓ $n'(t) = 28.3606e^{0.55t}$ ✓ Determine when $n'(t) = 250\ 000$ $250\ 000 = 28.3606e^{0.55t}$ ✓✓ $8815.05 = e^{0.55t}$ ✓ $t = \frac{\ln(8815.05)}{0.55}$ ✓ $t = 16.5168$ ✓ years ✓	**8c.** comprehend information requires use of derivative function generate equation to solve using given information use appropriate method to determine time (include units in years)
	Question 9 (SF 4 marks)  Points of intersection are (0.662, 1.89) ✓✓ and (1.55, 4.76) ✓✓	

9. use technology to determine both points of intersection	Bounded area = $\int_{.662}^{1.55}(-3e^{-x}+x^2+3)-(2e^x-3x)dx$ ✓✓ Bounded area = $\int_{0.662}^{1.55}\left(-3\cdot e^{-x}+x^2+3-\left(2\cdot e^{x}-3\cdot x\right)\right)dx$ 0.298649 Bounded area = 0.298649 ✓✓	
use technology to solve for bounded region communicate method used		
	## Question 10 (CU 8 marks) Sketch functions to identify area:  ✓✓ Line contains $(0,\ 0)$ and x intercepts of parabola at points $(0,\ 0)$ and $(2,\ 0)$	

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10.
translate information into mathematically workable format by

- sketching
- identifying half the area equates to half the area under the parabola

use procedure for determining definite integrals to determine the area under the parabola and therefore determine half the area

use procedure to determine m in terms of intersection point a

use procedure to determine a using the area between the curves

solve for a

solve for equation of the line

evaluate the reasonableness of the solution

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Total area = $\int_0^2 2x(2-x)dx$✓

$= 4x^2 - \frac{2x^3}{3}\Big\rvert_0^2 = 8 - \frac{16}{3} = \frac{8}{3}$✓

∴ half of the area $= \frac{4}{3}$ ✓ (technology may be used here)

Let $y = mx$ and $y = 2x(2-x)$ intersect at $x = a$✓

Then $ma = 2a(2-a)$✓

$m = 4 - 2a$✓

Hence $y = (4-2a)x$

$\int_0^a (4x - 2x^2) - (4-2a)x\,dx = \frac{4}{3}$✓✓

$\int_0^a -2x^2 + 2ax\,dx = \frac{4}{3}$✓

$\rightarrow \frac{-2a^3}{3} + \frac{(2a)a^2}{2} = \frac{4}{3}$✓

$\rightarrow \frac{-4a^3 + 6a^3}{6} = \frac{4}{3}$

$2a^3 = 8$

$a = \sqrt[3]{4}$✓

(Note: Both the approximate solution and use of technology to determine a solution are acceptable.)

e.g.

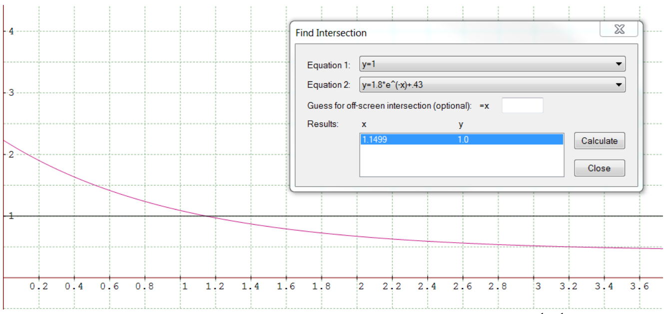
$\mathrm{nSolve}\left(\int_0^a \left(4\cdot x - 2\cdot x^2 - (4 - 2\cdot a)\cdot x\right)dx = \frac{4}{3}, a\right)$

1.5874

The equation of the line is $y = (4 - \sqrt[3]{4})x$✓

Use technology to determine the area to verify the line divides the area in half. ✓✓

Question 11 (CU 7 marks)



✓✓

Given the run terminates at a place 1 kilometre above sea level:

∴ determine x value when $y = 1$ ✓

$x = 1.1499$ ✓

Equation for curve vertically 2 metres (0.002 kilometres)

$H_2 = 1.8e^{-x} + 0.43 + 0.002$

$= 1.8e^{-x} + .432$ ✓✓

Cross-sectional area using integration ✓

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11. comprehend concept to decide on method of solution: • graph function • translate information into a mathematical representation generate translated curve and/or identify area between the two curves as the cross-sectional area recall rule for determining cross-sectional area solve for the volume of snow on the run communicate using mathematical symbols and conventions (e.g. units) represent ideas in a way that makes sense — relate parts in an orderly, consistent way (justifying procedures)	Area $= \int_0^{1.1499} (H_2 - H)dx$ $= \int_0^{1.1499} 0.002dx$ ✓✓ $= 0.0023$ ✓ Volume = area × width $= 0.0023 \times .3$ ✓ $= 0.00069$ km^3 ✓ (Communication of response ✓✓)	