About patterns and algebra

Algebra is often viewed as an abstract and symbolic component of the mathematics curriculum; however, algebraic thinking begins as soon as students notice consistent change and seek to describe it. For example, in the early years, algebraic thinking can be represented through everyday situations such as balancing concrete materials using balance baskets. This progresses to the use of more symbolic representations in the upper levels when letters are used to generalise thinking or to consider situations using variables.

The Patterns and Algebra strand supports thinking, reasoning and working mathematically. Students have to extend their thinking beyond what they see to generalise about situations involving unknowns.

This strand draws together the fundamental properties and relationships that guide arithmetic thinking to algebraic thinking. It involves the development of the knowledge, procedures and strategies associated with two topics:

- Patterns and functions, which develops understandings of consistent change and relationships
- Equivalence and equations, which develops understandings of balance and the methods associated with solving equations.

Algebraic notation enables us to represent problems in the form of equations (number sentences) that involve unknown quantities and to solve them efficiently.

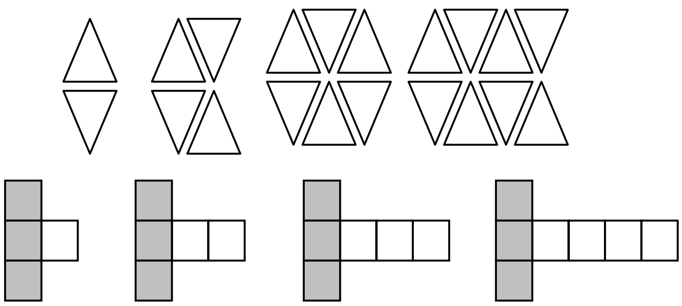
Patterns

Patterns are an important focus in the early stages of the development of algebraic thinking. The patterns in algebra fall into two broad categories: repeating patterns and growth patterns.

A **repeating pattern** is defined as a pattern in which there is a discernible unit of repetition — a cyclical structure that can be generated by the repeated application of a smaller portion of the pattern. The following example is a repeating pattern:

red blue green red blue green

Growth patterns have discernible units commonly called terms and each term in the pattern depends on the previous term and its position in the pattern. For example:

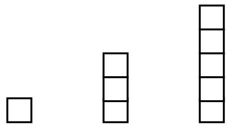


Number patterns are constructed of numbers but are generally limited to those patterns in which the numerical value of the elements is important. For example, 1 2 2 1 2 2 1 2 2 is not considered a number pattern whereas 3 7 11 15 19 is a number pattern. Number patterns, therefore, are growth patterns.

In the early years of schooling, students identify, describe and create repeating and growing patterns that use spatial materials and involve colour, shape and size as well as patterns based around the use of the body and senses (claps, dance movements, musical and speaking tones, and rhythms). These patterns will be based on a repeating part or a growing part and students should identify them as repeating patterns or growing patterns. They copy, continue and then translate patterns to create similar ones reading ‘along the grain’ of the pattern.

As students discern the 'rules' for repeating patterns such as two clicks of the fingers then a clap, they can translate that rule into a different representation of the same pattern — for example, two stamps of a foot and a touch of the shoulders. To represent a growing pattern, such as adding a tile to the stem of the T for each term (as shown on the previous page), students could use the rule to translate the pattern using other shapes, musical tones or movements.

Once students have consolidated their understandings of rules, they can be challenged to translate visual, auditory and movement growing patterns into number patterns. It is growing patterns, not repeating patterns, that translate into number patterns. For example:



1 3 5

When working with the rule, teachers should encourage students to think, reason and work mathematically by asking questions that extend students' thinking beyond what is seen in the patterns provided or what can be readily developed using more materials.

All of the patterns in the examples below are based on the repetition of three elements and could be translations of each other:

A B C A B C A B C

□ ◊ ⌂ □ ◊ ⌂ □ ◊ ⌂

clap stamp click clap stamp click clap stamp click.

As students' understandings develop, and they analyse and interpret repeating and growing patterns, the focus shifts towards higher-order thinking activities. These activities could be seen as reading 'across the grain' activities where students are asked to link elements or terms of the pattern with their position in the pattern. For example, to extend thinking and reasoning about the following pattern,

A B C A B C A B C A B C …

students could be asked the following questions:

- What element would be in the 20th position in the pattern? What element would be in the 200th position? How do you know?
- Where would the 30th 'A' be positioned in the pattern?
- What rule could you use for working out the position of any of the 'A' elements?
- If the pattern consisted of 50 elements, how many As, Bs and Cs would be used?

The questions above encourage thinking and reasoning beyond a description of the repeating parts of the pattern. This approach can be used with any repeating pattern to ensure that students understand patterns.

When thinking about position, it may assist students if they view the patterns with the position marked below the elements under analysis. For example:

Element	A	B	C	A	B	C	A	B	C	A	B	C	…	…
Position	1	2	3	4	5	6	7	8	9	10	11	12	13	…

The rules determining the position of each element might be easier to see:

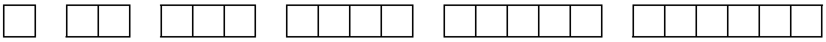
- 'C' has positions 3, 6, 9, 12, … (multiples of 3)
- elements 'A' and 'B' have different starting positions, but then '+ 3' to those starting positions each time
- 'A' has positions 1, 4, 7, 10, 13, … and 'B' has 2, 5, 8, 11, …

These position numbers become growth patterns that students can analyse and interpret.

Other patterns will be based on growth where there is a discernible change in size as the pattern proceeds. *Growth* patterns involve a change between successive elements in the pattern, and this change can be described using an arithmetic rule.

While students may continue the growth as soon as they analyse the pattern and work out the rule, they need to be confident about the nature of these patterns and be able to create rules and patterns by themselves. They need to distinguish between repeating and growth patterns, and notice differences in the rules and relationships that belong to the various patterns.

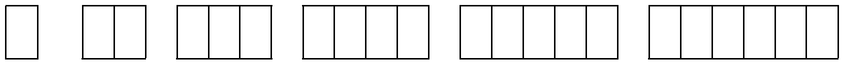
The patterns below are different growth patterns:



2 4 6 8 10 12 14 …(+ 2)

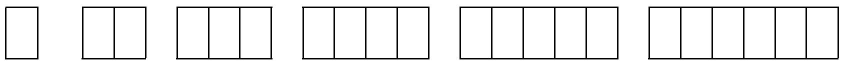
1 3 6 10 15 21 28 …(+ 2, + 3, + 4, + 5 …).

When students are analysing growth patterns they should think and reason about position and how to work out elements in any position. With the pattern below, the growth is in relation to spatial aspects, but this can be viewed in different ways.

Element						
Position	1	2	3	4	5	6

If the focus is on the number of rectangles, there is a simple direct relationship between position and rectangles — the 10th position would have 10 rectangles; if there were 15 rectangles, this element would be in the 15th position in the pattern.

A different rule applies, however, if the focus is placed on the number of lines or concrete materials, such as straws or matchsticks required to construct each element in the pattern. It could be said that position 1 needs 4 lines, position 2 needs 3 more (7 altogether), position 3 needs 3 more (10 altogether), and so on as summarised below.

Element						
Position	1	2	3	4	5	6
Number of lines						

Students then reason that the 20th element would be made up of the 4 original lines and then 19 lots of 3 sticks added on (one less than the position in focus). The specific rule would read:

number of lines used in 20th element = $4 + (3 \times 19)$.

Students should be encouraged to reason about other rules that lead to the same result. Some may reason that the rule should be based on the actual position of the element. With that in mind, the rule could be adapted to begin with 1 line.

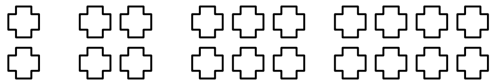
The alternative rule for the 20th element would read:

number of lines = $1 + (3 \times 20)$.

Students should compare the rules and explain why both work well. The next stage is to generalise the rule so the number of lines needed to complete the element in any position can be calculated:

number of lines = $1 + (3 \times n)$.

Before being introduced to formal notation systems, students should be encouraged to describe the relationship between the term and its position in everyday language. For example:



1 2 3 4

Some possible descriptions are:
For each position there are twice as many crosses as the position number.
There are two rows and each row has as many crosses as the position number.

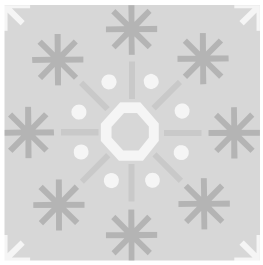
A real-life example of such patterns occurs when buying postage stamps. If each stamp is 50c, the pattern for buying an undefined number of stamps is as follows:

Number of stamps	1	2	3	4	5	6
Cost	50c	$1.00	$1.50	$2.00	$2.50	$3.00

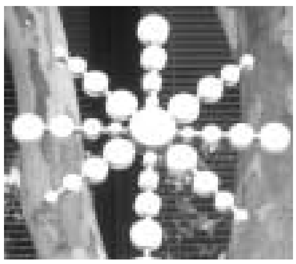
The ‘across the grain’ activity would involve asking students: *How much do 23 stamps cost?* or *If I bought $10.00 worth of stamps, how many would I get?*

Circular and *radiating* patterns are other types of repeating patterns and are also investigated in this topic. These are spatial patterns.

Circular patterns comprise concentric circles of shapes that may completely cover a surface in the same way that tiles are arranged on a tabletop.



Radiating patterns consist of arms that radiate out from a central point. The rule applies to each of the radiating arms.



Patterns can be represented in a variety of ways, but the most common is the sequenced listing of elements. When the position of each element in the pattern is also displayed (as shown the on previous page), other representations can also be explored. These include tables of ordered pairs and graphs that can be used to make predictions about further elements in the pattern. The table of ordered pairs that follows illustrates the growth pattern represented by the rectangles.

Number of rectangles	1	2	3	4	5	6	7	8	9	10	11	12
Number of lines	4	7	10	13	16	19	22	25	28	31	34	37

Functions

Function refers to the mathematical relationship between two values — the second value depends on (is a function of) the first one. The idea of *functional relationships* is important in describing relationships in the real world and in areas of mathematics where variables represent different things. For the stamp-purchasing example mentioned earlier, the cost depends on the number of stamps being bought. This can be expressed in language as: the total cost is the number of stamps multiplied by 50 cents (the cost of each stamp) or, 'to work out the total cost, you multiply the number of stamps by 50 cents'. This could also be expressed as:

total cost = number of stamps x 50 cents

Another example relates to interest earned on savings in a bank account. The amount of interest earned is a function of the interest rate (which can be viewed as a constant) and the account balance. This can be expressed as:

interest rate x account balance = amount of interest received

A further example occurs when the amount charged by a tradesperson for a service call depends on (is a function of) the time spent providing the service and the hourly rate charged.

In measurement, there is a relationship between the size of the unit being used to measure an attribute and the number of units required. There is also a functional relationship between the radius and perimeter of a circle.

Once students have an understanding of these functional relationships, symbols become an efficient way of representing relationships — a way of clarifying concepts at an abstract level. They progress to representing relationships with both discrete and continuous data, and model these relationships in more sophisticated ways by using linear and non-linear models.

When developing early understandings of functions, students represent relationships concretely, verbally, pictorially, electronically, symbolically or with combinations of these. They might write a story or talk about an illustration that shows the amount of money in a school building fund over time where the amount depends on (is a function of) the amount of time the fund has been established. Students could also sketch an informal graph that shows how the temperature changes during the day where the temperature depends on (is a function of) the time of the day.

Students should be provided with opportunities to experiment with simple function machines where they can create a rule to use or identify the rule being used. To assist students to identify the rule, the results of the function machine could be recorded in a table. For example, where the results from an experiment are:

Input	Output
1	4
2	5
3	6
4	7

the rule is 'add 3'. The rule of a relation can be written as an equation. For the above example, the rule would be written as: $x + 3 = n$ where 'x' is the input and 'n' is the output. It could also be written using arrow notation as:

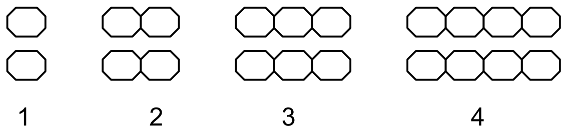
$$x \xrightarrow{+\,3} \text{'}n\text{'}$$

In arrow notation, the diagram is read in the direction of the arrow, thus x changes by the addition of 3 to give n.

An important dimension of functional thinking is backtracking, or using the inverse rule or inverse function to solve unknowns. For the example above, the inverse rule is 'subtract 3'. Therefore, if we know that the output is 23, we can work out the input by simply subtracting three. The inverse rule can be represented as:

$$x \xleftarrow{-\,3} \text{'}n\text{'} \qquad \text{or} \qquad x = n - 3$$

Growth patterns are also represented as functions. For example:



can be expressed as:

Position	Number of tiles
1	2
2	4
3	6
4	8

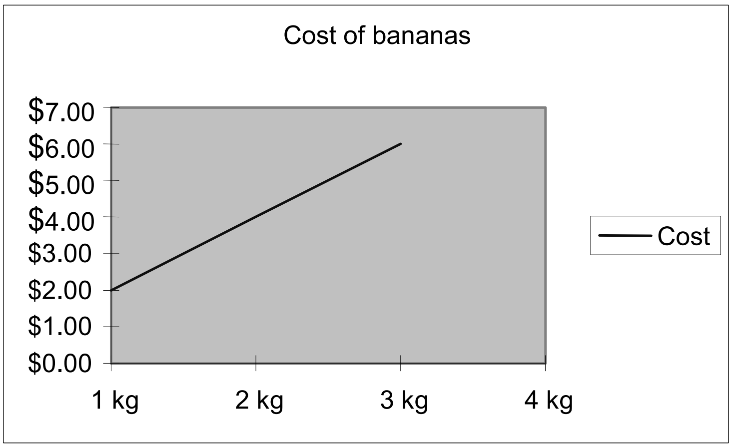
Here the function rule is ‘multiply by 2’ and the inverse rule is ‘divide by 2’. Therefore, the number of tiles in the 50th position is 100 (multiply by 2) and the position that has 240 tiles is 120 (divide by 2).

Other experiences could include gathering or generating data involving the relationship between input and output, such as the cost of increasing quantities of fruit (e.g. 1 kg of bananas costs $2.00; 2 kg cost $4.00) or the length of a shadow at different times of the day. When working with such data, students learn that functions are things that can vary (variables) and therefore have a changing relationship with other variables: changes to one variable result in changes in another.

Other representations of relationships include rules, ordered pairs and graphs.

Quantity	Cost
1 kg bananas	$2.00
2 kg bananas	$4.00
3 kg bananas	$6.00

The example above could be represented graphically to encourage students to think, reason and work mathematically beyond what is represented to make generalisations about the cost of larger quantities not depicted in the graph based on the function. Using a graphical representation gives students an indication of the trend of situations and enables them to have an expectation of an answer to a question using the rule that can be either confirmed or refuted.



Equivalence and equations

The development of the notion of equivalence is a key focus in the early levels of the topic, *Equivalence and equations*. A major conceptual understanding required by students is that the equals sign represents a relationship between the numbers on each side of the sign. A misconception commonly held by students is that the equals sign requires students to carry out an operation or that the number to the right of the equals sign is the answer. For example, in the equation:

$2a + 6 = 20$, the answer is not 20

and for the equation: $3 + 4 = 5 + 2$, the answer is not 5.

Students should be encouraged to think in terms of equivalence and balance when confronted by an equals sign. The relational thinking expressed by the use of the equals sign is crucial in the development of computation skills and algebraic thinking. Students need this understanding to engage with number sentences such as:

$$5 + 6 = 5 + 5 + 1.$$

In the early years, students use concrete materials to help them develop the concepts of equivalence and balance. Balancing devices such as bucket balance or balance scales allow students to manipulate objects to demonstrate equivalence. This also helps them understand the conservation of number — the number of objects remains the same when they are rearranged spatially. The equation also remains balanced when the same action is applied to each side, for example, adding or multiplying each side by the same number.

Students use the language of equivalence such as ‘equal to’, ‘same as’, ‘not equal to’, and ‘different from’ as they add, subtract or rearrange the objects until the scales balance. In this way they are encouraged to find more than one equivalent expression. For example, for the equation above students might find that:

$5 + 6$ is equal to $5 + 5 + 1$, or $5 + 4 + 2$, or $3 + 3 + 3 + 2$.

Concrete materials and balancing devices are also used to help students find missing addends. For example, when there are 5 square blocks and 6 round blocks on one side of a scale, and 5 square blocks and 4 round blocks on the other side, to answer the question ‘What do I have to do to balance the scale?’ students add 2 round blocks to one side, or take 2 round blocks from the other side. They can also apply this understanding when finding the unknown for simple equations, such as:

$\square + 3 = 7$

This equation remains balanced when 3 is subtracted from both sides.

Therefore, $\square = 4$

This is called the *balance strategy* for finding the unknown. The question that students need to ask is:

> What do we need to do to isolate the unknown while remembering to keep the equation balanced?

This thinking can be extended to find the unknown for more complex equations such as:

$2 \times \square + 8 = 30$

The thinking required to solve this equation is: First take 8 from both sides then divide both sides by 2. Thus the equation becomes:

$2 \times \square = 22$ (taking 8 from both sides)

Then, when each side is divided by 2, it becomes:

$\square = 11$

For this example, students need to understand three ideas:

- an equation remains balanced if we perform the same operation to each side
- subtracting 8 is the inverse of adding 8
- dividing by 2 is the inverse of multiplying by 2.

The use of concrete materials also assists in the development of the understanding that expressions, which are equal to the same thing, are equal to each other (transitive relation).

Students progress to using the symbols > (greater than) and < (less than) to demonstrate what is happening with the balance before they use the symbol ≠ (does not equal) to represent inequality. Students should be taught strategies other than those for solving expressions to help them decide which symbol to use. For example, to decide whether 13 + 20 x 3 is ‘equal to’, ‘greater than’ or ‘less than’ 13 x 20 + 3, students may consider the quantity of the numbers, the operations included in the equation or estimation strategies.

Students can use concrete materials to represent equations; however, as they progress to using symbolic representations, they begin to use shapes or other symbols to represent unknowns. When teaching students about equivalence and equations, it is helpful to use realistic situations that can be represented pictorially. Students are then able to move from pictorial to symbolic representations and see that algebra is the representation of a problem.

The use of arrow diagrams to represent equations also leads to the use of more formal notation. For example, when working out catering costs, students could represent the problem as follows:

number of hamburgers x \$2 → + cost of packets → total cost.

Students may represent an ‘unknown’ with a shape before using a letter as the representation. It is important to adhere to the convention of using different shapes to represent unknowns. This means that where the same shapes are used, they have the same value. Where different shapes are used, they may have the same or different values. For example,

$$8 - \square = \square$$

has only one solution. That is, the unknown is 4 because the same shape must have the same value. By contrast:

$$\triangle + \square = 8$$

has nine solutions including 4 and 4 because different shapes can have the same value. A real-life example of this problem is: *Boys and girls can go to the movies. If we have 8 movie tickets, how many boys and how many girls can go?*

As equations become more complex and more than one operation is involved, the advantages of using letters instead of shapes and symbols become obvious. This will happen only when students have a sound understanding of patterns, functions and equivalence.

Students develop the understanding that a letter may be used to identify an ‘unknown’ quantity or ‘variable’. The fact that letters have no intrinsic or unique value may be a source of confusion. If ‘a’ is equal to 1 in one equation, some students are under the misconception that ‘a’ always equals 1 and need to be shown that the value of a letter is variable.

It is important to change the symbols so students focus on what it is being represented rather than on the symbol. The use of letters in this way will act to avoid confusion for some students who have become familiar with the use of letters in other mathematical situations. In measurement, for example, letters represent standard units — 5 L represents 5 litres — and in formulae, letters represent various attributes — L for length, W for width, and C for circumference.

A thorough understanding of the importance of the order of operations needs to precede exposure to formal algebraic expressions, especially the use of brackets. Where there is no context to indicate an order in an equation of mixed operations, the correct order is to complete any operations that appear in brackets, followed by multiplication and division before addition and subtraction. The equation is then solved by working from left to right.

The understanding of number properties and the development of algebraic thinking are dependent on a sound understanding of the set of numbers (identity elements) that, when combined with another number in an operation, leave that number unchanged. The identity element under addition is 0:

$$9 + 0 = 9.$$

The identity element under multiplication is 1:

$$9 \times 1 = 9.$$

This knowledge, together with the understanding that a number multiplied by 0 is an empty set (zero), supports the development of generalisations by students.

Students draw on their understandings of the identity element for addition and/or multiplication when solving equations.

Algebra is a way of thinking and reasoning about relationships. All strands of the *Years 1 to 10 Mathematics Syllabus* provide opportunities for the development of algebraic thinking. As students engage in investigations based on real-life situations, they use algebraic thinking and reasoning when they identify variables and the relationships between them, represent a situation using words, concrete materials, pictures, tables, symbols or graphs, and use the relationships to predict information or generate solutions.

Resources

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